

777b

E L E M E N T S
O F
G E O M E T R Y;

I N W H I C H
A L L T H E M A T E R I A L P R O P O S I T I O N S

I N
T h e f i r s t S i x , E l e v e n t h , a n d T w e l f t h B o o k s

O F
E U C L I D ,

A r e d e m o n s t r a t e d w i t h
C O N C I S E N E S S A N D P E R S P I C U I T Y :

B Y .
W I L L I A M S C O T T .

K

E D I N B U R G H :

P R I N T E D F O R C H A R L E S E L L I O T , E D I N B U R G H ;
A N D G . R O B I N S O N , L O N D O N .

M D C C L X X X I I .

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P R E F A C E.

TH E prolixity of Euclid's demonstrations has given rise to several modern systems of Geometry, some of which possess a considerable degree of merit. None of these, however, exhibit that conciseness and perspicuity, which the author conceived it possible to unite in a work of this nature, and which he has attempted in the following pages. His endeavours, he flatters himself, have not been unsuccessful.

THE substance of Euclid's Elements is here comprised in narrower bounds than in any book he has met with; many unnecessary propositions are excluded, and some of the principal obstacles removed. The demonstrations respecting *Proportion* and *Solids*, in particular, are, perhaps, as short and simple as any that can be devised;

iv P R E F A C E.

at the same time that they are founded on axioms, which, he presumes, must satisfy the most scrupulous*.

He has not the vanity, however, to think his work entirely faultless. In many respects, he is sensible, it is short of perfection. But, though objections may be made, and errors detected, he trusts the severity of criticism will not be employed against a performance, which was attended with peculiar difficulty, and of which the manifest design is, the assistance of youth, and the cultivation of useful science.

* See pages 54th and 101st.

A TABLE, shewing where the most important **THEOREMS** in **EUCLID'S ELEMENTS** are demonstrated in the following work.—*N. B.* The numbers on the left, denote the Propositions in Euclid; those on the right, denote the corresponding Propositions in these Elements, with the Sections in which they are to be found.

Book I.		Book II.		Book V.		Book XI.	
4.	- - - I. I.	1.	- - - 5. 2.	4.	- - - 4. 4.	2.	- - - 1. 6.
5, 6.	- - - 6. I.	4.	- - - 6. 2.	12.	- - - 2. 4.	3.	- - - 2. 6.
8.	- - - 11. I.	5.	- - - cor. 3. 6. 2.	15.	- - - cor ^s . to 2. 4.	4.	- - - 3. 6.
13, 14.	- - - 2. I.	12.	- - - 8. 2.	16, 18.	- - - 3. 4.	5.	- - - 4. 6.
15.	- - - cor. 5. 2. I.	13.	- - - 9. 2.	22.	- - - 5. 4.	6.	- - - 5. 6.
18, 19.	- - - 7. I.	Book III.		23.	- - - 6. 4.	8.	- - - 6. 6.
20.	- - - 9. I.			24.	- - - 1. 4.	9.	- - - 7. 6.
21.	- - - 10. I.			Book VI.		10, 15.	- - - 11. 6.
24, 25.	- - - 13. I.					13.	- - - cor. 1. 5. 6.
26.	- - - 5. I.	2.	- - - 5. 3.			14.	- - - 9. 6.
27. to 30.	- - - 3. I.	3.	- - - 4. 3.			16.	- - - 10. 6.
32.	- - - 4. I.	7, 8.	- - - 11. 3.	1.	- - - 1. 5.	17.	- - - 13. 6.
33.	- - - 3. 2.	14.	- - - 9. 3.	2.	- - - 3. 5.	18.	- - - 8. 6.
34.	- - - cor. 1. 1. 2.	15.	- - - 10. 3.	4, 5.	- - - 4. 5.	19.	- - - 12. 6.
35. to 38.	- - - 2. 2.	18.	- - - 6. 3.	6.	- - - 5. 5.	35.	- - - 15. 6.
39, 40.	- - - cor. 2. 1. 5.	20.	- - - 3. 3.	8.	- - - 7. 5.	Book XII.	
41.	- - - cor. 5. 2. 2.	31.	- - - cor. 2. 3. 3.	14, to 17.	- - - 2. 5.		
43.	- - - 4. 2.	32.	- - - 8. 3.	19, 20.	- - - 9. 5.		
47.	- - - 7. 2.	35.	- - - 12. 3.	22, 31. - cor ^s . to 9. 5.	- - - 10. 5.		
		36.	- - - 13. 3.	33.	- - - 10. 5.	2.	- - - 12. 5.

The substance of the numerous Theorems respecting Solids in the 11th and 12th Books, is contained in Section 7th.

The subject of logarithms is a subject of great importance in the study of mathematics and in the application of mathematics to the sciences.

IX		X		XI		XII	
1	10	1	10	1	10	1	10
2	20	2	20	2	20	2	20
3	30	3	30	3	30	3	30
4	40	4	40	4	40	4	40
5	50	5	50	5	50	5	50
6	60	6	60	6	60	6	60
7	70	7	70	7	70	7	70
8	80	8	80	8	80	8	80
9	90	9	90	9	90	9	90
10	100	10	100	10	100	10	100
11	110	11	110	11	110	11	110
12	120	12	120	12	120	12	120
13	130	13	130	13	130	13	130
14	140	14	140	14	140	14	140
15	150	15	150	15	150	15	150
16	160	16	160	16	160	16	160
17	170	17	170	17	170	17	170
18	180	18	180	18	180	18	180
19	190	19	190	19	190	19	190
20	200	20	200	20	200	20	200
21	210	21	210	21	210	21	210
22	220	22	220	22	220	22	220
23	230	23	230	23	230	23	230
24	240	24	240	24	240	24	240
25	250	25	250	25	250	25	250
26	260	26	260	26	260	26	260
27	270	27	270	27	270	27	270
28	280	28	280	28	280	28	280
29	290	29	290	29	290	29	290
30	300	30	300	30	300	30	300
31	310	31	310	31	310	31	310
32	320	32	320	32	320	32	320
33	330	33	330	33	330	33	330
34	340	34	340	34	340	34	340
35	350	35	350	35	350	35	350
36	360	36	360	36	360	36	360
37	370	37	370	37	370	37	370
38	380	38	380	38	380	38	380
39	390	39	390	39	390	39	390
40	400	40	400	40	400	40	400
41	410	41	410	41	410	41	410
42	420	42	420	42	420	42	420
43	430	43	430	43	430	43	430
44	440	44	440	44	440	44	440
45	450	45	450	45	450	45	450
46	460	46	460	46	460	46	460
47	470	47	470	47	470	47	470
48	480	48	480	48	480	48	480
49	490	49	490	49	490	49	490
50	500	50	500	50	500	50	500
51	510	51	510	51	510	51	510
52	520	52	520	52	520	52	520
53	530	53	530	53	530	53	530
54	540	54	540	54	540	54	540
55	550	55	550	55	550	55	550
56	560	56	560	56	560	56	560
57	570	57	570	57	570	57	570
58	580	58	580	58	580	58	580
59	590	59	590	59	590	59	590
60	600	60	600	60	600	60	600
61	610	61	610	61	610	61	610
62	620	62	620	62	620	62	620
63	630	63	630	63	630	63	630
64	640	64	640	64	640	64	640
65	650	65	650	65	650	65	650
66	660	66	660	66	660	66	660
67	670	67	670	67	670	67	670
68	680	68	680	68	680	68	680
69	690	69	690	69	690	69	690
70	700	70	700	70	700	70	700
71	710	71	710	71	710	71	710
72	720	72	720	72	720	72	720
73	730	73	730	73	730	73	730
74	740	74	740	74	740	74	740
75	750	75	750	75	750	75	750
76	760	76	760	76	760	76	760
77	770	77	770	77	770	77	770
78	780	78	780	78	780	78	780
79	790	79	790	79	790	79	790
80	800	80	800	80	800	80	800
81	810	81	810	81	810	81	810
82	820	82	820	82	820	82	820
83	830	83	830	83	830	83	830
84	840	84	840	84	840	84	840
85	850	85	850	85	850	85	850
86	860	86	860	86	860	86	860
87	870	87	870	87	870	87	870
88	880	88	880	88	880	88	880
89	890	89	890	89	890	89	890
90	900	90	900	90	900	90	900
91	910	91	910	91	910	91	910
92	920	92	920	92	920	92	920
93	930	93	930	93	930	93	930
94	940	94	940	94	940	94	940
95	950	95	950	95	950	95	950
96	960	96	960	96	960	96	960
97	970	97	970	97	970	97	970
98	980	98	980	98	980	98	980
99	990	99	990	99	990	99	990
100	1000	100	1000	100	1000	100	1000

quantity of logarithms is a subject of great importance in the study of mathematics and in the application of mathematics to the sciences.

A B B R E V I A T I O N S

C O N T E N T S.

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E R R A T A.

- Page 5. Line 25. No. 12. should be 11.
 23. 8. AM should be BM.
 70. 10. Cancel *opposite*.

From page 74th to page 84th inclusive, each number referring to the plates should be 1 less.

A B B R E V I A T I O N S.

$+$, *more*; the sign of addition. Thus $a+b$ denotes the sum of the quantities expressed by a and b .

$-$, *less*; the sign of subtraction. Thus $a-b$ denotes the difference of a and b ; the less quantity being placed after the sign.)

∞ , *difference*. Thus $a \infty b$ denotes the difference of a and b , whichever is the greater.

$=$, *equal to*; the sign of equality. Thus $a=b$ denotes that a is equal to b .

$\angle$, *greater than*; the sign of excess. Thus $a \angle b$ denotes that a is greater than b .

$\succ$, *less than*; the sign of defect. Thus $a \succ b$ denotes that a is less than b .

A number prefixed to any quantity, shews how often that quantity is repeated, or what part of it is taken. Thus, $2a$, $\frac{1}{2}a$, denote respectively, twice a , half a .

The contractions def. post. ax. conf. hyp. rect. sq. denote respectively, definition, postulate, axiom, construction, hypothesis, rectangle, and square.

In the references (except in Sect. 1.) the right hand number points out the Section; the next to that, the Proposition.

The numbers in parentheses refer to figures in the plates.

ELEMENTS

GEOMETRY.

SECTION I.

Of TRIANGLES.

DEFINITIONS.

1. **G**EOMETRY is that science which teaches the properties of magnitude, abstractly considered, under one or more of its dimensions, length, breadth, and thickness.
2. A *line* has length only, as A (1).—The extremities or bounds of a line are *points*.
3. A *surface* has length and breadth only, as B (2).—The bounds of a surface are *lines*.
4. A *solid* has length, breadth, and thickness, as C (3).—The bounds of a solid are *surfaces*.

5. A *straight line* lies evenly between its extremities, or has every-where the same direction, as A (1).
6. An *angle* is the opening or mutual inclination of two straight lines which meet together, as D (4).
7. A *right angle* is the angle formed by one straight line meeting another so as to make the angles on both sides equal, as ABC (5). Either of the lines which form a right angle, is said to be *perpendicular* to the other.—N. B. When there is only one angle at a point, it may be expressed by a single letter at that point: but, when there are more angles, any one of them is expressed by three letters; the middle letter shewing the *angular point*, or point of meeting; the other two, the lines which form the angle, sometimes called its *legs*.
8. An *obtuse angle* is greater than a right angle, as A (6).
9. An *acute angle* is less than a right angle, as D (4).
10. A *plane surface* lies evenly between its extremities, or is such, that any two points whatever being taken therein, the straight line between them lies wholly in the surface, as B (2).
11. A *triangle* is a plane surface, bounded by three straight lines, as B (7).

§ 1. GEOMETRY.

12. An *equilateral* triangle has all its sides equal, as B (7).
13. An *isosceles* triangle has two sides equal, as B or C (7 or 8).
14. A *scalene* triangle has all its sides unequal, as D (9).
15. An *equiangular* triangle has all its angles equal, as B (7).
16. A *right-angled* triangle has one right angle, as A (10).—The side opposite the right angle is called the *hypotenuse*.
17. An *obtuse-angled* triangle has one obtuse angle, as D (9).
18. An *acute-angled* triangle has all its angles acute, as C (8).
19. *Parallel* straight lines are such as are in the same plane, and which, tho' produced infinitely both ways, would never meet, as BC, DE (11).

POSTULATES.

1. **T**HAT any given straight line may be produced either way, at pleasure.
2. That, in the greater of two given straight lines, a part may be taken equal to the less.
3. That, from one given point to another, a straight line may be drawn.

4. That any given angle may be bisected by a straight line.
5. That, from any point in a given straight line, another straight line may be drawn, making therewith an angle equal to any given angle.
6. That, from any given point, a straight line may be drawn perpendicular to any given straight line.
7. That a straight line may be drawn parallel to any given straight line, through any point in the same plane *.

* N. B. These Postulates are referred to only in Section 1. The operations mentioned in them are effected in Section 8, without the assistance of those Theorems in which they are required.—Note also, that (unless the contrary be mentioned) all lines in any Proposition are supposed to be straight, and to be drawn in the same plane.

A X I O M S.

1. **E**VERY whole is equal to the sum of all its parts, and greater than any one of them.
2. Things which are equal to one and the same thing, or to equal things, are equal to one another.
3. If equal things be equally increased, diminished, multiplied, or divided, the results will be equal.

4. If unequal things be equally increased, diminished, multiplied, or divided, the results will be greater or less than one another, according as the unequals from which they are derived are greater or less than one another.
5. In any expression, instead of any quantity, its equal may be substituted.—If $A=B+C$ and $D=C$, then $A=B+D$.
6. In an expression of continued equality or inequality, the first quantity is equal to, greater, or less than, the last, according to the sign.—If $A=B=C$, then $A=C$; if $A<B<C$, then $A<C$; and, if $A>B>C$, then, also, $A>C$.—*N. B.* This conclusion is always to be understood when such expressions occur in the following demonstrations, though it be not formally mentioned.
7. Two different straight lines cannot be drawn from one point to another.
8. Magnitudes which, being applied to one another, coincide exactly, are equal.
9. All right angles are equal.
10. Perpendiculars (MP, NQ) let fall upon either of two parallels (DE) from the other (BC) are equal (12). For, if unequal, 'tis manifest the parallels would meet, if infinitely produced towards the less perpendicular; which is absurd.
11. That supposition is false, which tends to prove an impossibility.

THEOREM I.

IF two sides (AB, AC) and the included angle (A) of one triangle (ABC) be respectively equal to two sides (DE, DF) and the included angle (D) of another triangle (DEF), the two triangles are equal in every respect: that is, the sides opposite equal angles are equal, as also the angles opposite equal sides; and the area of, or space occupied by, the one triangle, is equal to the area of, or space occupied by, the other.—(12, 13.)

CONCEIVE triangle ABC applied to DEF , with the point A upon D , and the side AB upon DE .—Because $AB=DE$, the point B will fall upon E ; because angle $A=D$, the side AC will fall upon DF ; because $AC=DF$, the point C will fall upon F ; and, because the points B, C , fall upon E, F , the side BC will fall upon EF ^a.—Therefore, triangle $ABC=DEF$, in every respect^b.

^a Ax. 7. ^b Ax. 8.

THEOREM II.

IF, from any point (A) between the extremities of a line (BC) any number of lines (AD, AE) be drawn on the same side of that line; the angles (BAD, DAE, EAC) thereby formed, are, together, equal to two right angles.—And if, from the point (A) in which two lines (BA, AC) meet, any number of lines (AD, AE) be drawn on the same side of those lines, thereby forming angles (BAD, DAE, EAC) which, together, are equal to two right angles; the two first-mentioned lines (BA, AC) make one continued line (BC).—(14).

PART I. LET AM be perpendicular to BC^a.—Then, $BAD + DAE + EAC =$ the whole angular space at A^b $= BAM + MAC^b = 2$ right angles^c.

PART II. If you say that any line, NA, and not BA, makes with AC one continued line—Then, $NAD + DAE + EAC = 2$ right angles^d $= BAD + DAE + EAC^e$; which is impossible^b.—Therefore, BA, AC, form one continued line, BC^f.

^a Post. 6. ^b Ax. I. ^c Conf. ^d Part I. ^e Hyp. ^f Ax. II.

COR. 1. ALL the angles proceeding from a point (O) taken all around it, are, together, equal to four right angles (15): for, any continued line, MN, passing through O, makes the angles on each side equal to two right angles.

COR. 2. If, of two angles (BAD, DAC) formed as in the Theorem (14), one angle be right, the other must be right; if one be obtuse, the other must be acute; and if one be acute, the other must be obtuse; because both, together, are equal to two right angles.

COR. 3. Any two such angles (BAD, DAC) are, together, equal to any other two such angles, together; each pair being equal to two right angles.

COR. 4. If an angle of one pair of such angles be equal to an angle of another pair of such angles, the remaining angles are equal; the two pairs being equal. Hence,

COR. 5. The opposite angles (AOC, DOB) formed by the intersection of two lines (AB, CD) are equal (16): for, the two pairs of angles, AOC BOC, and DOB BOC, have BOC common.

COR. 6. If two lines (CO, DO) meeting at a point (O) in a line (AB) make two opposite angles (AOC, DOB) equal, these two lines (CO, DO) make one continued line (CD): for, since $DOB = AOC$, $DOB + COB = AOC + COB = 2$ right angles.

T H E O R E M III.

IF one line (AB) meeting two other lines (CD, EF) make the alternate angles (CGH, FHG) equal, these two lines are parallel.—(17).—And, if two lines (CD, EF) be parallel, the alternate angles (CGH, FHG) are equal.—(18).

PART I. IF you say, that CD, EF, will meet towards F, as at I, forming a triangle IHG; in GC (produced if necessary) take $GK=HI^a$, and join HK^b .—Then, in triangles KHG, IGH (because GK, GH, and angle CGH, are respectively equal to HI, HG, and angle FHG^c) angle KHG=DGH^d: whence (by adding the equal angles FHG, CGH) angle KHG+FHG=DGH+CGH^e=2 right angles^f. Therefore, KHI, as well as KGI, form one continued line^f; which is impossible^g. Therefore, CD, EF, cannot meet towards F^h. In the same manner, it might be shewn that they cannot meet towards C.—Therefore, CD, EF, are parallelⁱ.

PART II. If you say, that angles CGH, FHG, are unequal, let CGH be the greater; draw GM, making angle MGH=FHG^j; from G, let fall

^a Post. 2. ^b Post. 3. ^c Conf. and Hyp. ^d 1. ^e Ax. 3.
^f 2. ^g Ax. 7. ^h Ax. 11. ⁱ Def. 19. ^j Post. 5.

GP perpendicular to EF; and, from any other point, N, in GC, let fall NQ, perpendicular also to EF, and cutting GM in R^k.—Then, because $MGH = FHG^1$, GM is parallel to EF^m, and, consequently, $RQ = GP^n$. But CD is also parallel to EF^c; therefore, $NQ = GP^n$. Hence, $RQ = NQ^o$; which is impossible^p.—Therefore, angle $CGH = FHG^h$.

COR. 1. IF one line (AB) meeting two other lines (CD, EF) make the external angle (AGD) equal to the opposite internal angle (AHF) on the same side; or make the opposite external angles (AGD, BHE) equal; or make the two internal angles (DGH, FHG) on the same side, together, equal to two right angles; these two lines (CD, EF) are parallel: for, in any of these cases, it will readily appear, from the preceding Theorem, that angle $CGH = FHG$.—(17).

COR. 2. If two lines (CD, EF) be parallel, the external angle is equal to the opposite internal angle on the same side; the opposite external angles are equal; and the two internal angles on the same side, are, together, equal to two right angles: for, angle $CGH = FHG$; from which these equalities will readily be derived by Theor. 2.

COR. 3. Perpendiculars (GP, NQ) to the same line (EF) are parallel to one another; because angle $GPF = NQF$.—(18).

^k Post. 6. ^l Conf. ^m Part I. ⁿ Ax. 10. ^c Hyp.
^o Ax. 2. ^p Ax. 1. ^h Ax. 11.

COR. 4. A line (NQ) perpendicular to either of two parallel lines (EF) is also perpendicular to the other (CD); because angle $DNQ = EQN$, and $CNQ = FQN$.

COR. 5. Lines (A, B) parallel to the same line (C) are parallel to one another; because angle $D = E = F$.—(19).

COR. 6. If the legs (AB, CB) of one angle (B) be respectively parallel to the legs (DE, FE) of another angle (E), these two angles are equal: for, producing the legs (BC, ED) till they meet in a point, O, we have angle $B = BOE = E$.

THEOREM IV.

THE three angles (CBA, BCA, BAC) of any triangle (ABC) are, together, equal to two right angles.—(21).

THROUGH A, draw DE parallel to BC^a.—Then, $CBA = DAB$, and $BCA = EAC$ ^b. Therefore, $CBA + BCA + BAC = DAB + EAC + BAC$ ^c = 2 right angles^d.

COR. 1. THE external angle (ACF) formed by producing any side of a triangle, is equal to both the internal opposite angles (ABC, BAC).

^a Post. 7. ^b 3. ^c Ax. 3. ^d 2.

together, and consequently is greater than either of them : for, $ACF + ACB = 2 \text{ right angles} = ABC + BAC + ACB$; whence, $ACF = ABC + BAC$.

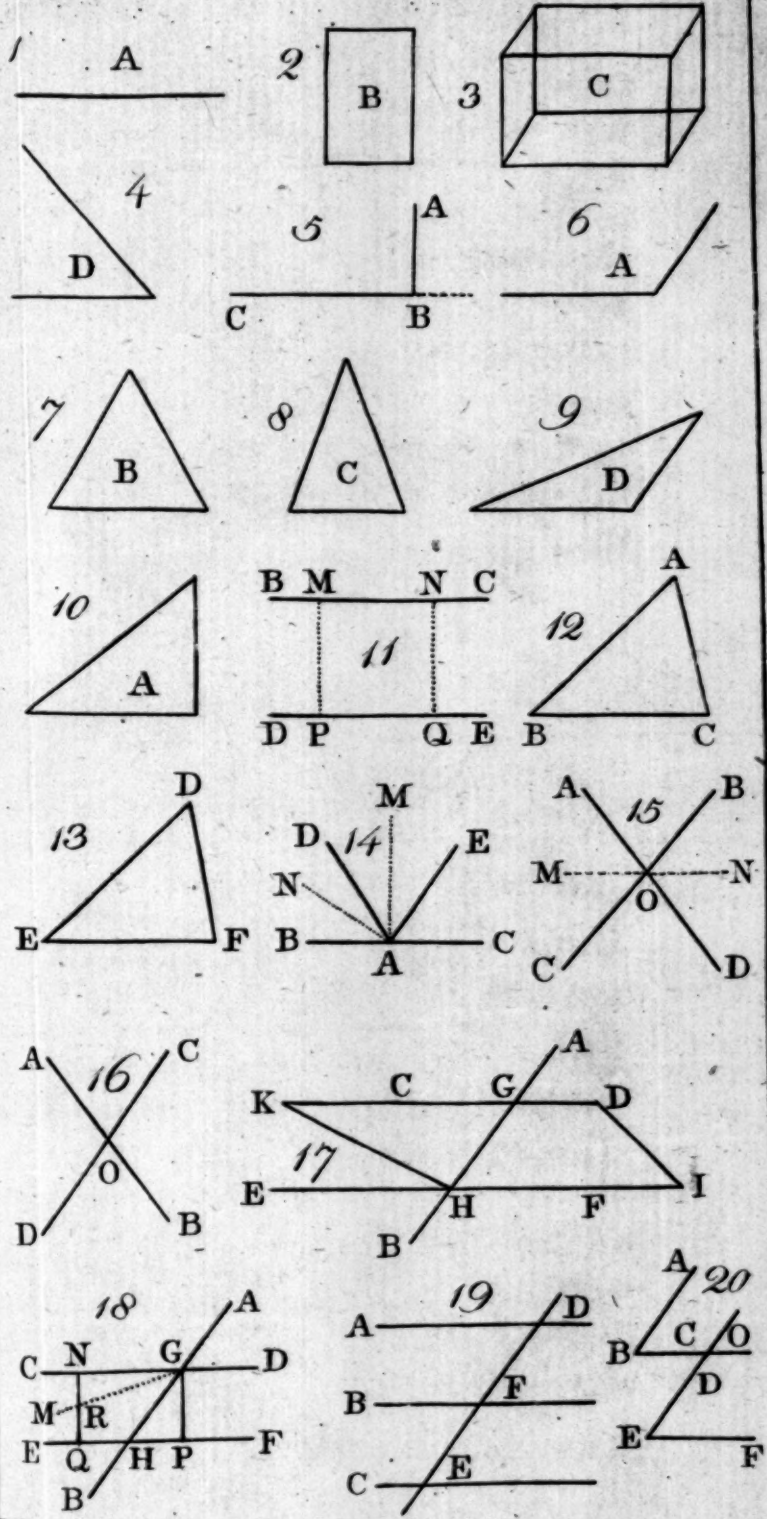
COR. 2. Any two angles of a triangle, are, together, less than two right angles ; consequently, there can be only one right or obtuse angle in a triangle : the two least angles of every triangle are acute ; and, from the same point, to the same line, there can be drawn only one perpendicular.

COR. 3. If one angle of a triangle is right, the other two are, together, equal to a right angle ; and, if two angles of a triangle are, together, equal to a right angle, the remaining angle is right.

COR. 4. Any angle of a triangle is right, obtuse or acute, according as it is equal to, greater, or less than the sum of the remaining angles.

COR. 5. The three angles of any triangle, are, together, equal to the three angles of any other triangle, together : whence, if one angle of one triangle be equal to one angle of another triangle, the sum of the remaining angles of the one, is equal to the sum of the remaining angles of the other ; and, if two angles of one triangle, separately or together, be equal to two angles of another triangle, separately or together, the remaining angles are equal.

COR. 6. The sum of the internal angles of any straight-lined figure or surface (O) is equal to twice as many right angles as the figure has sides,





abating four right angles (22). For, drawing lines from the angular points, to any point (O) within the figure, 'tis manifest, that the sum of the internal angles of the figure is equal to the sum of the angles of all the triangles thereby formed, abating the angles at O; or equal to twice as many right angles as there are triangles, or as there are sides in the given figure ^a, abating four right angles ^b.

COR. 7. The sum of the external angles of any straight-lined figure (O) is equal to four right angles, being equal to what the internal angles of the figure want of twice as many right angles as the figure has sides ^c.

THEOREM V.

IF the three angles (A, B, C) of one triangle (ABC) be respectively equal to the three angles (D, E, F) of another triangle (DEF) and two sides (BC, EF) opposite equal angles also equal, the two triangles are equal in every respect.—(23, 24).

If you say, that AB, DE, are unequal, let DE be the greater; take $GE = AB$ ^d, and draw FG ^e. Then, in triangles GEF, ABC, (because $GE = AB$ ^f, $EF = BC$ ^g, and angle $E = B$ ^g) angle

^a 4. ^b Cor. 1. 2. ^c 2. ^d Post. 2. ^e Post. 3. ^f Conf. ^g Hyp.

$GFE = C^a = DFE^b$; which is impossible^c. Therefore, $AB = DE^d$; and, consequently, triangle $ABC = DEF$, in every respect^a.

COR. If two triangles have two angles equal to two, and two sides equal whether between or opposite equal angles, the triangles are equal in every respect^a.

THEOREM VI.

IF two sides (AB , AC) of a triangle (ABC) be equal, the angles (B , C) opposite those sides, are equal.—And, if two angles (B , C) of a triangle (ABC) be equal, the sides (AB , AC) opposite those angles, are equal.—(25).

DRAW AD , bisecting angle BAC^e .

PART I. IN triangles ABD , ACD , (because angle $BAD = CAD$, $AB = AC$, and AD common ^f) angle $B = C^a$.

PART II. IN triangles ABD , ACD (because AD is common, angle $B = C$ and angle $BAD = CAD^f$) $AB = AC$.

COR. I. EVERY equilateral triangle is equiangular, and every equiangular triangle is equilateral.

^a 1. ^b Hyp. ^c Ax. I. ^d Ax. II. ^e Post. 4.

^f Conf. and Hyp. ^g Cor. 5. 4. ^h Cor. 5.

COR. 2. In an isosceles triangle, the angles opposite the equal sides are both acute; and, if the remaining angle be right, they are, each, half a right angle.

COR. 3. In an equilateral or equiangular triangle, each of the angles is one-third of two right angles, or two-thirds of one right angle.

COR. 4. A straight line bisecting the vertical angle of an isosceles triangle, bisects the base, and is perpendicular to it: thus, $BD=CD$, and angle $ADB=ADC$.

T H E O R E M VII.

IN any triangle (ABC) the greater side subtends the greater angle, and the greater angle is subtended by the greater side.—(26).

PART I. IF $AC < AB$, angle $ABC < ACB$.—Take $AD=AC^a$, and draw DB^b . Angle $ABC < ABD^c$. But angle $ABD=ADB^d$. Therefore, angle $ABC < ADB^e < ACB$.

PART II. If angle $ABC < ACB$, $AC < AB$.—If you say that AC is equal to or less than AB , then must angle ABC be equal to or less than ACB^f , which is impossible^h. Therefore, $AC < AB^i$.

^a Post. 2. ^b Post. 3. ^c Ax. I. ^d 6. ^e Ax. 5. ^f Cor. I. 4.
^g 6. & Part I. 7. ^h Hyp. ⁱ Ax. II.

THEOREM VIII.

OF all the lines (OP, OQ, OR, OS) that can be drawn from a point (O) to another line (AB) that is the least (OP) which is perpendicular to it; and, of the rest, any one (OQ or OR) is less than any other (OS) which is at a greater distance from the perpendicular.—(27).

1. BECAUSE angle $OQP > OPQ^a$, $OP > OQ^b$.

2. Because angle $OSP > OPS^a > OQS^c$, $OQ > OS^b$.

3. In PS, take $PQ = PR^d$, and draw OQ^e . Then, $OQ = OR^f$. But $OQ > OS$ (by the preceding case); and, consequently, $OR > OS^g$.

COR. If a line (OP) drawn from a point (O) to another line (AB) be the least that can be drawn to it from that point, the two lines are perpendicular to one another.

^a Cor. 2. 4. ^b 7. ^c Cor. 1. 4. ^d Post. 2. ^e Post. 3.

^f 1. ^g Ax. 5.

T H E O R E M IX.

ANY two sides (AB, AC) of a triangle (ABC) are, together, greater than the third side (BC).—(28).

IN CA produced ^a, take AD = AB ^b, and join DB ^c.—Then, angle ABD = D ^d. But, angle CBD < ABD ^e. Therefore, angle CBD < D ^f; and, consequently, CD (or AB + AC) > BC ^g.

THE same may also be proved from Theor. 8. by letting fall a perpendicular from A upon BC.

T H E O R E M X.

IF, of two triangles (ABC, DBC) having a side (BC) common, the one triangle wholly include the other, the two remaining sides (AB, AC) of the one, are, together, greater than the two remaining sides (DB, DC) of the other; but the angle (A) contained by the former sides, is less than the angle (BDC) contained by the latter.—(29, 30).

^a Post. 1. ^b Post. 2. ^c Post. 3. ^d 6. ^e Ax. 1. ^f Ax. 5. ^g 7.

CASE 1. When the point D is in a side of the greater triangle (29) — Then, $AB + AD < DB^a$; and, consequently, $AB + AD + DC$ (or $AB + AC$) $< DB + DC^b$. But angle $A > BDC^c$.

CASE 2. When the point D is within the greater triangle (30). — Produce BD till it meet AC in E^d. Then, $AB + AC < EB + EC^e < DB + DC^e$. But angle $A > BEC^e > BDC^e$.

THEOREM XI.

IF the three sides (AB, AC, BC) of one triangle (ABC) be respectively equal to the three sides (DE, DF, EF) of another triangle (DEF) the two triangles are equal in every respect. — (31, 32).

If you say, that angles CBA, FED, are unequal, let CBA be the greater; make angle CBG = FED^f, GB = DE^g = AB^h, and join GCⁱ. So will GC = DF^j = AC^h. Now, because AB + AC does not exceed GB + GC, triangle ABC cannot wholly include triangle GBC^k. The point G must therefore fall without. Join AG. Then (because GB = AB and GC = AC) angle BAG = BGA, and angle CAG = CGA^a. But angle BAG $<$ CAG^b: Therefore also, angle BGA $<$ CGA^c;

^a 9. ^b Ax. 4. ^c Cor. I. 4. ^d Post. 1. ^e Case 1.
^f Post. 5. ^g Post. 2. ^h Hyp. ⁱ Post. 3. ^j 1. ^k 10.

which is impossible^b. Angles CBA, FED, are therefore equal^d; and, of consequence, triangle $ABC = DEF$ in every respect^e.

THE same may be proved from Theorems 1. and 6. by making angle $FEH = ABC$, $HE = AB$, and drawing HF , HD : for it will thence appear, that triangle $ABC = HEF = DEF$ in every respect.

THEOREM XII.

IF two sides (AB , AC) of one triangle (ABC) be respectively equal to two sides (DE , DF) of another triangle (DEF) and two angles (B , E) opposite equal sides also equal, the two triangles are equal in every respect, provided the angles (ACB , DFE) opposite the other equal sides, be either both acute or both obtuse.—(33, 34; 35, 36).

CASE I. When angles ACB , DFE , are both acute (33, 34).—If you say that BC , EF , are unequal, let BC be the less; in BC produced, take $BG = EF$ ^f, and join AG ^g. Then $AG = DF$ ^e $= AC$ ^b. Therefore, angle ACG is acuteⁱ; which is impossible^j.

^a 6. ^b Ax. 1. ^c Ax. 5. ^d Ax. 11. ^e 1. ^f Post. 2.
^g Post. 3. ^h Hyp. ⁱ Cor. 2. 6. ^j Cor. 2. 2.

CASE 2. When angles ACB , DFE , are both obtuse (35, 36).—If you say that BC , EF , are unequal, let BC be the greater; take $BG=EF^a$, and join AG^b . Then $AG=DF^c=AC^d$. Therefore, angle ACG is acute^e; which is impossible^d.

Therefore, in both cases, $BC=EF^f$; and, consequently, triangle $ABC=DEF$ in every respect^g.

COR. RIGHT-ANGLED triangles (ABC , DEF) having equal hypotenuses (AC , DF) and two other sides (AB , DE) equal, are equal in every respect; because angles B , E , are equal^h, and angles C , F , both acuteⁱ.—(37, 38).

THEOREM XIII.

IN any two triangles (ABC , DEF) having two sides (AB , AC) of the one equal respectively to two sides (DE , DF) of the other, the greater of the included angles (A) is subtended by the greater of the remaining sides (BC)—and the greater of the remaining sides (BC) subtends the greater of the included angles (A).—(39, 40, 41, 42, 43).

^a Post. 2. ^b Post. 3. ^c 1. ^d Hyp. ^e Cor. 2. 6. ^f Ax. II.
^g 1. or II. ^h Ax. 9. ⁱ Cor. 2. 4.

PART I. IF angle $A < EDF$, $BC < EF$.—
Make angle $EDG = A^a$, $DG = AC^b = DF^c$, and
draw EG^d .

1. If EG fall upon EF (40), $EG < EF^e$.
2. If EG fall below EF (41), $DG + EG < DF + EF^f$: whence (by subtracting the equals DG , DF^g) $EG < EF^h$.
3. If EG fall above EF (43), join FG^d . Then, because $DG = DF^g$, angle $DFG = DGF^i$. But angle $EFG < DFG^e$. Therefore, angle $EFG < DGF^j < EGF^k$; and, consequently, $EG < EF^l$.

In every case, therefore, $EG < EF$. But (because $DE = AB^e$, $DG = AC^g$, and angle $EDG = A^a$) $EG = BC^l$. Therefore, in every case, $BC < EF$.

PART II. If $BC < EF$, angle $A < EDF$.—If you say that angle A is equal to or less than EDF , then must BC be equal to, or less than EF^m ; which is impossible^e. Therefore, angle $A < EDF^n$.

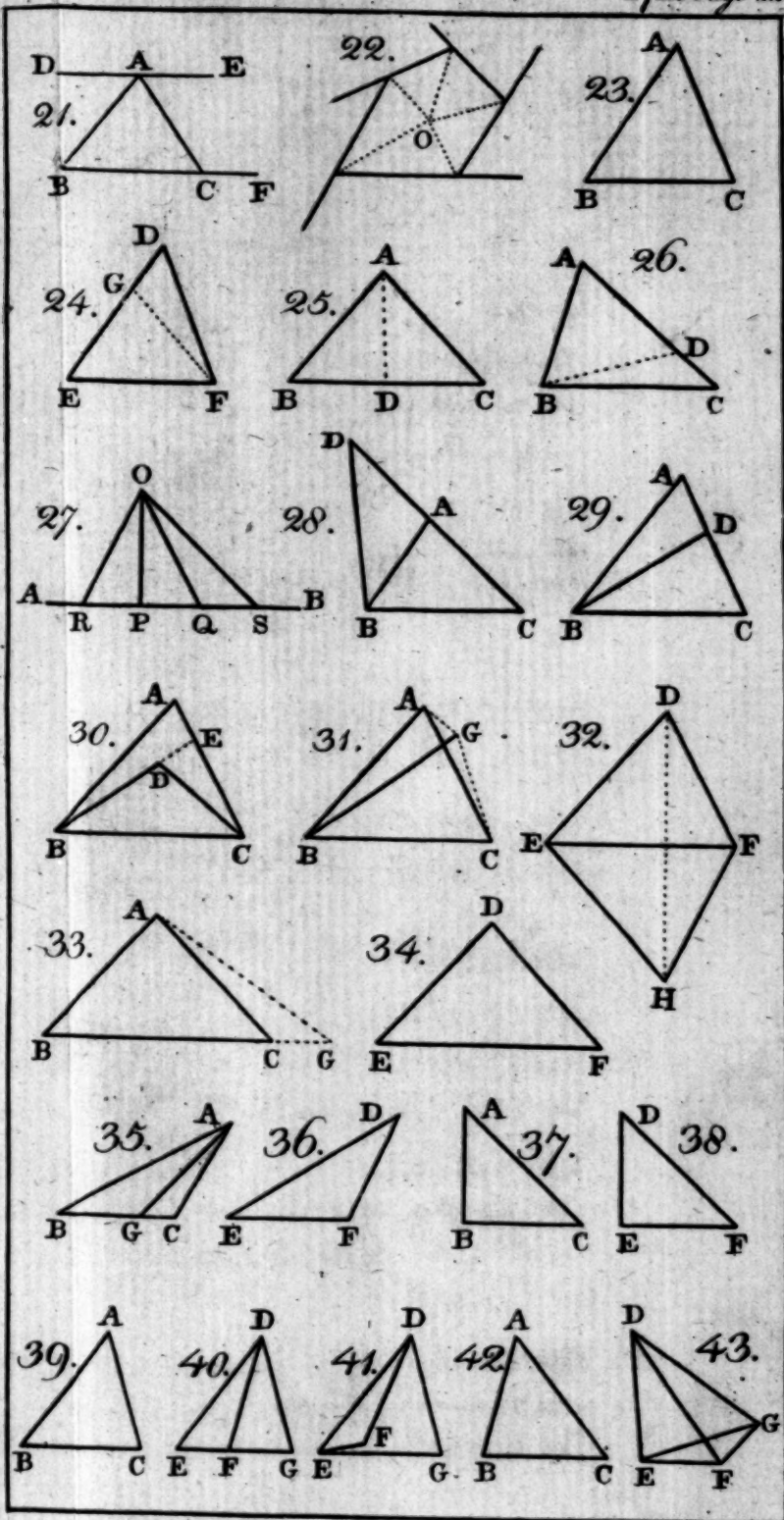
^a Post. 5. ^b Post. 2. ^c Hyp. ^d Post. 3. ^e Ax. 1. ^f 10.
^g Conf. ^h Ax. 4. ⁱ 6. ^j Ax. 5. ^k 7. ^l 11. ^m 1. and
Part I. 13. ⁿ Ax. 11.

SECTION II.

Of PARALLELOGRAMS.

DEFINITIONS.

1. **A** *Quadrangle* or quadrilateral is a plane surface bounded by four straight lines, as AD (44).—*N. B.* It is sometimes pointed out by the letters at all the angles, but more generally by two letters at opposite angles.
2. A *parallelogram* is a quadrangle, of which the opposite sides are parallel, as AD (45).
3. A *rectangle* is a parallelogram, of which all the angles are right angles, as AD (46).
4. A *square* is a parallelogram, of which all the angles are right angles, and all the sides equal, as AD (47).
5. The *diagonal* of a quadrangle is a straight line drawn from one of two opposite angles to the other, as BC (48).
6. The *complements* of a parallelogram are two parallelograms formed on different sides of the diagonal, by two straight lines which are drawn parallel to the sides, and intersect each





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other and the diagonal in the same point; as AO, DO (48).

7. The *base* of a parallelogram or triangle, is any side thereof upon which it is supposed to stand; as CD (49).

8. The *altitude* of a parallelogram or triangle, is a straight line drawn perpendicular to the base from the opposite angle; as AM (49).

N. B. A rectangle is said to be contained under the two lines which are the base and altitude thereof: thus, rectangle AD (46) is said to be contained under AC, CD, and, for brevity, may be called rectangle AC, CD.

T H E O R E M I.

IF two sides (AB, AC) and the included angle (A) of one parallelogram (AD) be respectively equal to two sides (EF, EG) and the included angle (E) of another parallelogram (EH), the two parallelograms are equal in every respect. (50, 51):

DRAW the diagonals BC, FG.—The triangle CAB=GEF in every respect^a. But (because angle ABC=DCB^b, angle ACB=DBC^b, and BC

^a 1. 1. ^b 3. 1.

common) triangle $CDB = CAB$ in every respect^a; and (for similar reasons) triangle $GHF = GEF$ in every respect. Therefore, triangle $CDB = GHF$ in every respect^b; and, consequently, the whole parallelogram $AD =$ the whole parallelogram EH in every respect^c.

COR. 1. THE diagonal of a parallelogram divides it into two triangles, which are equal in every respect. Triangle $CDB = CAB$ — Therefore,

COR. 2. The opposite sides and opposite angles of a parallelogram are equal. Consequently,

COR. 3. If one angle of a parallelogram be right, all the angles are right; being, together, equal to four right angles.

COR. 4. Rectangles contained under equal lines are equal.

COR. 5. Equal lines have equal squares.

THEOREM II.

PARALLELOGRAMS (CB , GF) having equal bases (CD , GH) and equal altitudes (AM , EN) are equal in area, or occupy equal spaces (52, 53).—*N. B.* This is all that is to be understood by equal surfaces, unless it be otherwise expressed.

^a Cor. 5. 1. ^b Ax. 2. ^c Ax. 3.

ON CD, GH, produced, let fall the perpendiculars BP, FQ.—Then (because $AC=BD$ ^a and $AM=BP$ ^b) triangle $AMC=BPD$ ^c, which being subtracted severally from the whole figure ABPC, there remains rect. MB=par. CB^d. In the same manner rect. NF=par. GF. But rect. MB=rect. NF^e. Therefore also par. CB=par. GF^f.

COR. 1. TRIANGLES (CAD, GEH) having equal bases (CD, GH) and equal altitudes (AM, EN) are equal; being halves of equal parallelograms (CB, GF).

COR. 2. Every parallelogram is equal to a rectangle having the same (or an equal) base and altitude.

COR. 3. Every triangle is equal to half a rectangle or parallelogram, having the same or an equal base and altitude.

COR. 4. Parallelograms (or triangles) having the same or equal bases, and being between the same parallels, are equal; because their altitudes are also equal. Hence,

COR. 5. Every triangle is equal to half a parallelogram, having the same or an equal base, and being between the same parallels.

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^a Cor. 2. I. 2. ^b Ax. 10. ^c Cor. 12. I. ^d Ax. 3.

^e Cor. 4. I. 2. ^f Ax. 2.

THEOREM III.

IF, in any quadrangle (AD) two opposite sides (AB, CD) be equal and parallel; or, if both pairs of opposite sides (AB CD, and AC BD) be equal; or, if both pairs of opposite angles (BAC BDC, and ABD ACD) be equal; the quadrangle is a parallelogram.—(50).

DRAW the diagonal BC.

PART I. IF AB be equal and parallel to CD.—Then, in triangles ABC, DCB (because AB=CD^a, BC common, and angle ABC=DCB^b) angle ACB=DBC^c. Therefore AD is a parallelogram^b.

PART II. IF AB=CD and AC=BD.—Then, in triangles ABC, DCB (because AB=CD, AC=BD^a, and BC common) angle ABC=DCB, and angle ACB=DBC^d. Therefore AD is a parallelogram^b.

PART 3. IF angle BAC=BDC, and angle ABD=ACD.—Then (because angle BAC=BDC^a) angle ABC+ACB=DCB+DBC^e. But angle ABC+DBC=DCB+ACB^a; whence (by adding equals) 2 ABC+ACB+DBC=2 DCB+ACB+

^a Hyp. ^b 3. I. ^c I. I. ^d II. I. ^e Cor. 5. 4. I.

DBC^a ; whence, again, (by subtracting equals) $2\ ABC = 2\ DCB^a$; whence also (by equal division) angle $ABC = DCB^a$, which, taken from the equals ABD , ACD , leaves angle $DBC = ACB^a$. Therefore AD is a parallelogram^b.

THEOREM IV.

THE complements (AO , DO) of any parallelogram (AD) are equal.—(48).

FROM the equal triangles BAC , BDC^c , take the equals BGO , BFO^c , and the equals CEO , CHG^c , and there remains $AO = DO^a$.

THEOREM V.

THE rectangle (AD) contained under any two lines (AC , CD) is equal to the sum of all the rectangles contained under either of these lines (AC) and all the parts (CE , EF , FD) of the other.—(54).

FROM the points E , F , draw EG , FH , perpendicular to CD , and meeting AB in the points

C 2

^a Ax. 3. ^b 3. I. ^c Cor. I. I. 2.

G, H.—Then, CG, EH, FB, are rectangles under the whole line AC (for GE, HF, are each equal to AC^a) and the parts CE, EF, FD, of the other line CD^b; and $AD = CG + EH + FB$ ^c.

COR. 1. IF any number of rectangles have one side the same in each, the sum of all these rectangles is equal to one rectangle under that common side, and the sum of all the other sides under which they are contained.

COR. 2. If a line be divided into any number of parts, the sum of the rectangles contained under the whole line and the several parts, is equal to the square of the whole line.

THEOREM VI.

IF a line (AB) be divided into any two parts (AC, BC), the square of the whole line is equal to the squares of both the parts, together with twice a rectangle under the parts.—(55).

LET AE be the square of AB; take AF=AC, and draw FG, CH, parallel respectively to AB, AD.—Then, $AE = AI + EI + BI + DI$ ^c. But AE is the square of AB; and (because of parallels) AI is the square of AC, EI the square of BC, and BI, DI, rectangles under AC, BC^d. Therefore $ABsq. = ACsq. + BCsq. + 2 \text{ rect. } AC, BC$.

^a Ax. 10. ^b Conf. ^c Ax. 1. ^d Cor. 2. 1. 2.

COR. 1. THE square of any line is equal to four times the square of half that line.

COR. 2. The sum of the squares of two lines is equal to the square of their sum, abating twice a rectangle under the given lines.

COR. 3. The difference of the squares of two lines, is equal to a rectangle under the sum and difference of those lines. $AB^2 - AC^2 = BC^2 + 2 \text{ rect. } AC, BC = \text{rect. } AB, BC + \text{rect. } AC, BC = \text{rect. under } AB + AC \text{ and } BC \text{ (or } AB - AC)$.

COR. 4. The square of either of two lines is equal to the square of their sum, abating the square of the other line and twice a rectangle under the given lines.

COR. 5. Equal squares have equal sides; because unequal lines (AB, AC) have unequal squares.

T H E O R E M VII.

THE square of the hypotenuse (AB) of any right-angled triangle (ABC) is equal to the sum of the squares of the two other sides (AC, BC).—(56).

LET AE, AG, and BI, be the squares of AB, AC, and BC. Draw CL parallel to AD or BE;

C 3

^a Cor. I. 5. 2.

and join AH, BF, CD, CE. Then (CL being parallel to AD^a) triangle ACD = $\frac{1}{2}$ AL^b; and (because BG is one continued line^c parallel to AF^d) triangle ABF = $\frac{1}{2}$ AG^b. But (since AD = AB^d, AC = AF^d, and angle CAD = BAF^e) triangle ACD = ABF^f. Therefore $\frac{1}{2}$ AL = $\frac{1}{2}$ AG^g; and consequently AL = AG^e. In the same manner, BL = BI. Therefore AE = AG + BI^e; or, ABsq. = ACsq. + BCsq.

The same may be demonstrated various other ways.—Thus, let triangle ABC (57) be right-angled at C. In CA produced, take AD = BC; let CE be the square of CD; take FG, EH, each = BC; and draw BG, GH, HA.—Triangles ABC, BGF, GHE, HAD, are equal in every respect^f, and are, therefore, together, equal to 4 ABC, or 2 rect. AC, BC^h. From the equality of these triangles, it also appears, that AG is the square of ABⁱ. Hence (and because AD = BC) ABsq. + 2 rect. AC, BC, = CDsq. = ACsq. + BCsq. + 2 rect. AC, BC^k. Consequently, ABsq. = ACsq. + BCsq.^l

COR. 1. The square of either of the sides including a right angle in a triangle, is equal to the difference of the squares of the hypotenuse and the other side.

COR. 2. The difference of the squares (or a rectangle under the sum and difference) of the two sides (AB, BC) of any triangle (ABC) is

^a Conf. ^b Cor. 5. 2. 2. ^c 2. 1. ^d Hyp. ^e Ax. 3.

^f 1. 1. ^g Ax. 2. ^h Cor. 3. 2. 2. ⁱ 2. and 4. 1. ^k 6. 2.

^l Ax. 3.

equal to the difference of the squares (or a rectangle under the sum and difference) of the lines (AD, CD) intercepted by the extremes of the other side (AC) and the perpendicular (BD) let fall upon it from the opposite angle.—(58, 60). For $ABsq. = ADsq. + BDsq.^a$, and $BCsq. = CDsq. + BDsq.^a$. Whence $ABsq. \propto BCsq. = ADsq. \propto CDsq.^b$.—Consequently, also, rect. under AB + BC, $AB \propto BC, = \text{rect. under } AD + CD, AD \propto CD.^c$

T H E O R E M VIII.

THE square of a side (AB) subtending an obtuse angle in a triangle (ABC) is greater than the sum of the squares of the two other sides (AC, BC) by twice a rectangle under either of these sides (AC) and the line (CD) intercepted between the obtuse angle and a perpendicular (BD) let fall upon that side from the opposite angle.—(58).

$ABsq. = BDsq. + ADsq.^a$. But $ADsq. = CDsq. + ACsq. + 2 \text{ rect. } AC, CD.^d$. Therefore (by substitution) $ABsq. = BDsq. + CDsq. + ACsq. + 2 \text{ rect. } AC, CD.^e$. But $BDsq. + CDsq. = BCsq.^a$. Whence again (by substitution) $ABsq. = ACsq. + BCsq. + 2 \text{ rect. } AC, CD.^e$.

^a 7. 2. ^b Ax. 3. ^c Cor. 3. 6. 2. and Ax. 2. ^d 6. 2.

^e Ax. 5.

THEOREM IX.

THE square of a side (AB) subtending an acute angle in any triangle (ABC) is less than the sum of the squares of the two other sides (AC, BC) by twice a rectangle under either of these sides (AC) and the line (CD) intercepted between the acute angle and a perpendicular (BD) let fall upon that side from the opposite angle.— (59, 60, 61).

CASE 1. WHEN BAC is a right angle (59).
 $AB^2 = BC^2 - AC^2$.^a $= AC^2 + BC^2 - 2 AC^2$. (or 2 rect. AC, CD).

CASE 2. When the perpendicular falls within the triangle (60). $AB^2 = AD^2 + BD^2$.^b But $AD^2 = AC^2 - CD^2 - 2$ rect. AD, CD^c; and $BD^2 = BC^2 - CD^2$.^a Whence (by substitution) $AB^2 = AC^2 + BC^2 - 2 CD^2 - 2$ rect. AD, CD^d $= AC^2 + BC^2 - 2$ rect. AC, CD^e.

CASE 3. When the perpendicular falls without the triangle (61). $BC^2 = AB^2 + AC^2 + 2$ rect. AC, AD^f. Whence, $AB^2 = BC^2 - AC^2 -$

^a Cor. 1. 7. 2. ^b 7. 2. ^c Cor. 4. 6. 2. ^d Ax. 5.

^e Cor. 1. 5. 2. ^f 8. 2.

2 rect. AC, $AD^a = ACsq. + BCsq. - 2 ACsq. -$

2 rect. AC, $AD = ACsq. + BCsq. - 2 \text{ rect. AC,}$
 $CD^b.$

COR. FROM this and the two preceding Theorems, it is manifest, that any angle in a triangle is right, obtuse, or acute, according as the square of the side subtending it is equal to, greater, or less than, the sum of the squares of the sides which contain it.

^a Ax. 3. ^b Cor. I. 5. 4.

SEC.

SECTION III.

Of CIRCLES.

DEFINITIONS.

1. **A** *Circle* is a plane surface, bounded by one line called its *circumference*, which is every-where equally distant from a point within the circle called its *center*.—Thus AB is a circle, of which ACBDA is the circumference, and M the center.—(62).
2. The *diameter* of a circle, is a straight line passing through the center, and terminated both ways by the circumference; half of which, or a straight line from the center to the circumference, is called the *radius*.—Thus AB is the diameter, and AO or OC the radius of the circle AMC.—(63).
3. An *arch* of a circle, is any part of the circumference, as PMQ.—(63).
4. A *chord* or *subtense* of an arch, is a straight line joining the two extremities of the arch, as PQ.—(63).
5. A *segment* of a circle, is part of the circle contained under an arch and its chord, as

PMQP (63).—When the chord is a diameter, the segment is called a *semi-circle*, as AMBA.—(63).

6. A *sector* of a circle, is part of the circle contained under an arch and two radii drawn to the extremities of the arch, as AODA (63).—When the angle contained by the radii is a right angle, the sector is called a *quadrant*, as AOCA.—(63).
7. An *angle in a segment*, is the angle formed by two straight lines drawn from any point in the arch of the segment to the extremities of the chord, as angle ABC.—(64).
8. A line, or circle, is said to *touch* a circle, when they meet in a single point without cutting one another: thus AB touches the circle CD, which touches the circle DE. (65).
9. A *tangent* is a line touching a circle, and terminated at the point of contact; as AC or BC.—(65).

N. B. The *distance* of two points from one another, is the straight line joining them; and the distance of a point from a line, is a perpendicular from the point to the line.

THEOREM I.

CIRCLES (ABD, EFG) having equal diameters, or equal radii, are equal—and equal circles (ABD, EFG) have equal diameters, and equal radii.—(66, 67).

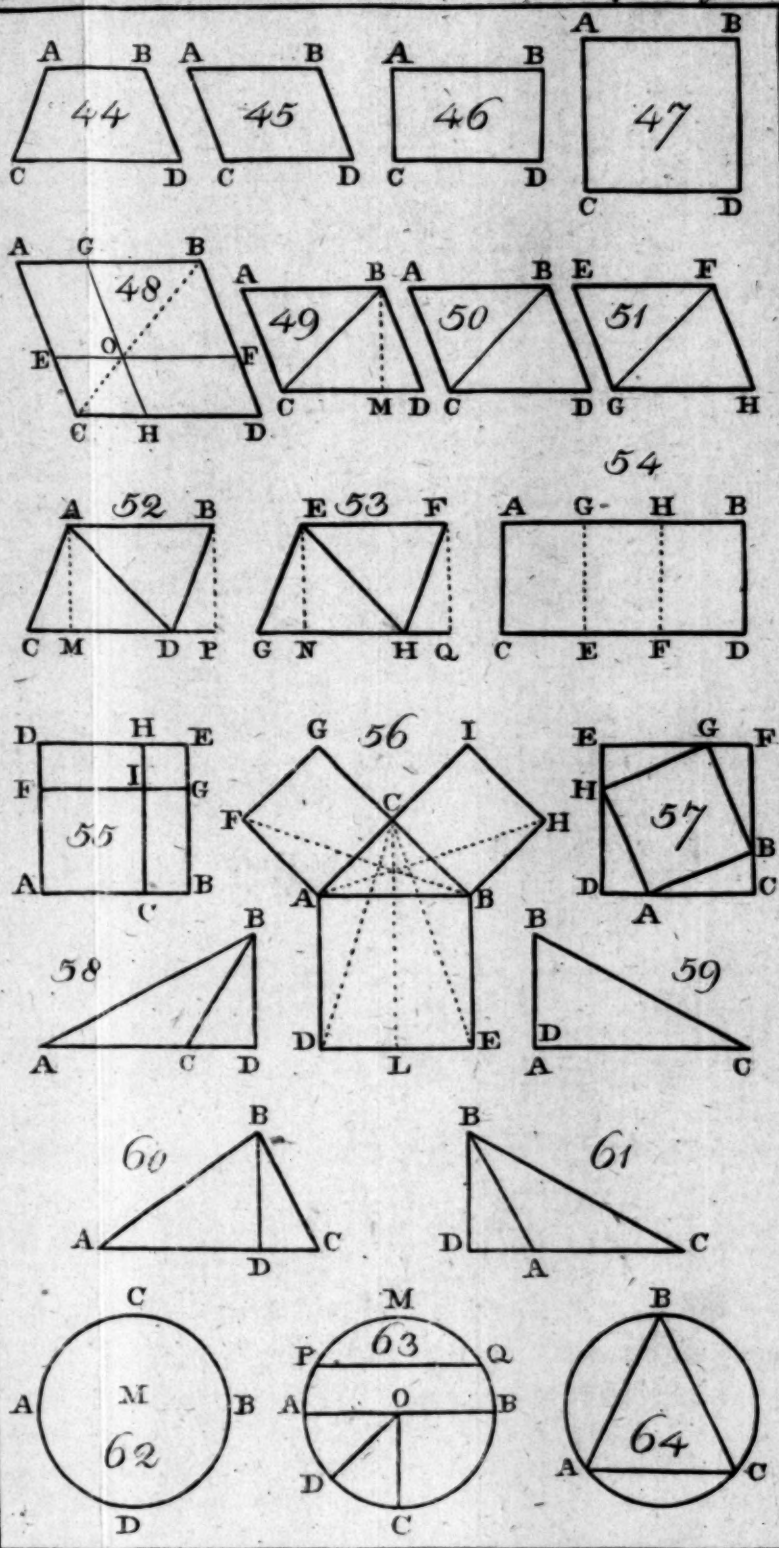
IN both cases, if ABD be conceived to be applied to EFG with the center M upon the center N, 'tis manifest the circumferences must coincide^a. Consequently, in the former case, the circles are equal^b; in the latter, the diameters and radii are equal^a.

THEOREM II.

IN the same or equal circles (ABD, EFG) equal angles at the center, or equal arches, form equal sectors (AMD, BMC, ENG).—(66, 67).

CONCEIVE the sectors applied to one another, so as that the equal angles or equal arches coincide; 'tis manifest the whole sectors must coincide, because the radii are equal^c: Therefore the sectors are equal^b.

^a Def. 1. 3. ^b Ax. 8. ^c Def. 1. & Theor. I. 3.



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COR. 1. QUADRANTS, and, consequently, Semi-circles, of the same or equal circles, are equal.

COR. 2. Equal angles at the center of the same or equal circles, stand upon, or are subtended by, equal arches and equal chords.

COR. 3. All angles at the center of the same or equal circles, standing upon equal arches or chords, are equal.

COR. 4. In the same or equal circles equal chords cut off equal segments, and subtend equal arches: also, equal arches are subtended by equal chords.

COR. 5. An angle at the center of a circle is right, obtuse, or acute, according as the arch it stands upon is equal to, greater, or less than a fourth part of the circumference.

THEOREM III.

AN angle (AOB) at the center of a circle is equal to twice an angle (ACB) at the circumference standing upon the same arch (AB).—(68, 69, 70).

CASE I. WHEN AC passes through the center (68).—Angle $AOB = ACB + OBC^a = 2 ACB^b$.

D

^a Cor. I. 4. I. ^b 6. I.

CASE 2. When AC, BC, are on different sides of the center (69).—Drawing the diameter CD, angle $AOD = 2 \angle ACD$ and angle $BOD = 2 \angle BCD^a$; whence, angle $AOB = 2 \angle ACB^b$.

CASE 3. When AC, BC, are on the same side of the center (70).—Drawing the diameter CD, angle $BOD = 2 \angle BCD$ and angle $AOD = 2 \angle ACD^a$; whence, angle $AOB = 2 \angle ACB^b$.

COR. 1. AN angle (ABC) at the circumference of any circle (71) standing upon any arch, is equal to an angle at the center standing upon half that arch (CD). For, drawing BD, AO, and CO, angle $ABD = \frac{1}{2} \angle AOD^a = \frac{1}{2} \angle COD^c$. But angle $CBD = \frac{1}{2} \angle COD^a$. Therefore, angle $ABC = \angle COD^b$.—Hence,

COR. 2. All angles at the circumference of the same or equal circles, standing upon equal arches or chords, are equal; because they are equal to equal angles at the center^c.

COR. 3. Equal angles at the circumference of the same or equal circles, stand upon equal arches and equal chords.

COR. 4. An angle at the circumference of a circle is right, obtuse, or acute, according as the arch it stands upon is equal to, greater, or less than half the circumference^d.—Therefore,

COR. 5. An angle in a semicircle, is right; an angle in a segment less than a semi-circle, is obtuse; and an angle in a segment greater than a semi-circle, is acute.

^a Case 1. ^b Ax. 3. ^c Cor. 3. 2. 3. and Ax. 2.

^d Cor. 5. 2. 3.

T H E O R E M IV.

A Line (AB) drawn from the center (A) of a circle to the middle of any chord (CD) is perpendicular to the chord. —And a line (BA) bisecting any chord (CD) at right angles, passes through the center of the circle. —(72).

PART I. DRAW AC, AD. In triangles ABC, ABD (because $AC=AD^a$, $BC=BD^b$, and AB common) angle $ABC=ABD^c$.

PART II. If you say any line BE, and not BA, passes through the center; then, angle $CBE=a$ right angle $d=CBA^b$, which is impossible e . Therefore BA passes through the center f .

COR. 1. SINCE from the same point to the same line there can be drawn only one perpendicular g , it follows that a perpendicular let fall from the center of a circle upon any chord bisects the chord; or it follows from Cor. to 12. 1.

COR. 2. Two chords crossing one another, cannot each bisect the other, unless they both pass through the center; because a line drawn from the center to the point of their intersection cannot be perpendicular to both.

COR. 3. A radius bisecting any chord, bisects also the arch; because angle $CAF=DAF$, arch $CF=DF^h$.

D 2

^a Def. 1. 3. ^b Hyp. ^c II. 1. ^d Part 2. ^e Ax. 1.

^f Ax. 11. ^g Cor. 2, 4. 1. ^h Cor. 2. 2. 3.

THEOREM V.

ANY chord (AB) of a circle falls entirely within the circle.—(73).

LET C be the center of the circle. Draw CA, CB; and, to any point, D, in AB, draw CD. —Then, because $CA = CB^a$, angle $A = B^b$. But angle $CDA < B^c$. Therefore, angle $CDA < A^d$; and, consequently, $CA < CD^e$. The point D, therefore, falls within the circle^a; that is, any point whatever in AB, and, consequently, AB itself.

COR. A line touching a circle in two points, cuts the circle: and a line touching a circle without cutting it, touches it only in one point.

THEOREM VI.

IF a line (AB) be perpendicular to the radius (CD) of a circle at the extreme point thereof (D), it touches the circle in that point.—And, if a line (AB) touch a circle in a point (D), it is perpendicular to the radius (CD) at that point.—(74).

^a Def. 1. 3. ^b 6. 1. ^c Cor. 1. 4. 1. ^d Ax. 5. ^e 7. 1.

PART I. FROM the center C, to any other point E, in AB, draw CE.—CE exceeds CD^a. Therefore, E is without the circle^b. Consequently, AB cannot touch the circle in E, that is, in any point except D.

PART II. Drawing CE as before, CD is less than CE^c, and, consequently, AB perpendicular to CD^d.

T H E O R E M VII.

IF the distance (AB) of the centers of two circles be equal to the sum or difference of their radii, the circles will touch one another in a point (C) in that line (AB).—(75, 76).

CASE 1. When AB=the sum of the radii (75)—Let AC be the radius of one circle; so is BC the radius of the other^c. Let DCE be perpendicular to AB. The circles touch DCE, and, consequently, one another, only in C^e.

CASE 2. When AB=the difference of the radii (76)—Let AC (AB produced) be the radius of the greater circle; so is BC the radius of the less^c. Take any other point, D, in the circumference of the less circle; and draw AD, BD.

D 3

^a 8. 1. ^b Def. 1. 3. ^c Hyp. ^d Cor. 8. 1. ^e 6. 3.

Because $BD = BC^a$, $AB + BD = AC^b$. But $AB + BD < AD^c$. Therefore $AC < AD^d$. The point D, therefore, is within the greater circle^a; and, consequently, the circles touch one another only in C.

COR. 1. If the distance of the centers of two circles be less than the sum of their radii, a part, at least, of the one circle will fall within the other; but if greater than the sum, the one circle will fall wholly without, and neither touch nor cut the other.—Also, if the distance of the centers of two circles be greater than the difference of their radii, a part, at least, of the one circle will fall without the other; but, if less than the difference, the one circle will fall wholly within, and neither touch nor cut the other. Therefore,

COR. 2. Two circles can touch one another only when the distance of their centers is equal to the sum or difference of their radii; in the former of which cases, they touch one another outwardly; in the latter, inwardly.—Whence, and from the Theorem,

COR. 3. If two circles touch one another, a line drawn through their centers, will pass thro' the point of contact; or, drawn from the point of contact to either center, will pass through the other center.

COR. 4. If the distance of the centers of two circles be less than the sum, and greater than the difference of their radii, those circles will cut each other; for part of the one circle will fall within, and part of it without the other^e.

^a Def. 1. 3. ^b Ax. 3. ^c 9. 1. ^d Ax. 5. ^e Cor. 1. 7. 3.

T H E O R E M VIII.

THE angles (ABC, ABD) formed by any chord (AB) and a line (DC) touching the circle at the extremity of the chord, are respectively equal to the angles (AEB, AFB) in the alternate segments.—(77).

LET BG be the diameter, and draw EG, FG. —Angles GBC, GBD, GEB, GFB, being right angles ^a, are all equal ^b; and ABG, AEG, AFG, being at the circumference, and standing upon the same arch, AG, are also equal ^c. To GBC, GEB, add ABG, AEG, and you have $ABC = AEB$ ^d. From GBD, GFB, take ABG, AFG, and you have $ABD = AFB$ ^d.

COR. 1. THE opposite angles of any quadrangle inscribed in a circle (that is, having its angular points in the circumference) are, together, equal to two right angles. $AEB + AFB = ABC + ABD = 2$ right angles.—Consequently,

COR. 2. If any side of a quadrangle inscribed in a circle be produced, the external angle thereby formed will be equal to the internal opposite angle. Because $AEB + AFB = 2$ right angles $= AFH + AFB$, $AEB = AFH$.—Hence also,

COR. 3. Every parallelogram inscribed in a circle is a rectangle ^e.

^a 6. 3. and Cor. 5. 3. 3. ^b Ax. 9. ^c Cor. 2. 3. ^d Ax. 3.

^e Cor. 2. I. 2.

THEOREM IX.

ANY two chords (AB, CD) at equal distances (OE, OF) from the center of a circle, are equal.—And equal chords (AB, CD) are at equal distances (OE, OF) from the center.—(78).

PART I. DRAW OB, OD.—Because triangles OBE, ODF, are right-angled at E, F^a, and have OB=OD^b and OE=OF^a, BE=DF^c; and, consequently, 2 BE or AB=2 DF or CD^d.

PART II. Draw OB, OD.—Because triangles OBE, ODF, are right-angled at E, F^a, and have OB=OD^b and BE or $\frac{1}{2}$ AB=DF or $\frac{1}{2}$ CD^d, OE=OF^c.

THEOREM X.

OF all chords which can be drawn in a circle, that one (AB) is the greatest which passes through the center (O); and, of the rest, any one (CD) is greater than any other which is at a greater distance from the center.—(79).

^a Hyp. ^b Def. 1. 3. ^c Cor. 12. 1. ^d Cor. 1. 4. and Ax. 3.

PART I. DRAW OC, OD.—Then, $OC + OD = OA + OB^a$. But $OC + OD < CD^b$; and, consequently, $OA + OB$ (or AB) $< CD^c$.

PART II. Draw radius OG perpendicular to CD at H; in which take OI, any distance greater than OH; through I, draw the chord EF, perpendicular to OG; and join OE, OF.—Then, in triangles COD, EOF, (because $OC = OE$, $OD = OF^d$, and angle $COD < EOF^e$) $CD < EF^f$, or greater than any chord at the same distance from the center as EF^g , that is, at a greater distance from the center than CD.

COR. A line greater than the diameter of a circle drawn from any point within the circle, will cut the circumference.

T H E O R E M X I.

IF, to the circumference of a circle, from any point (A) which is not the center, lines be drawn (80, 81, 82)—1. That line (AB) which passes through the center (O) is greater than any other (AC).—2. The remainder (AL) of the diameter (when the point is within the circle (80) is less than any other line (AC).—3. Any two lines (AC, AD) making equal angles with the

^a Def. 1. 3. and Ax. 3. ^b 9. 1 ^c Ax. 5. ^d Def. 1. 3.

^e Ax. 1. ^f 13. 1. ^g 9. 3.

greatest, are equal.—4. Any two (AD, AE, or AC, AE) making unequal angles with the greatest, are unequal, and that is the greater (AD or AC) which makes the less angle.—5. Of external lines (82) that one (AG) which produced would pass through the center, is less than any other (AH).—6. Any two (AH, AI) making equal angles with the least, are equal.—7. Any two (AI, AK, or AH, AK) making unequal angles with the least, are unequal; and that is the less (AI or AH) which makes the less angle.

DRAW OC, OD, OE, OH, OI, OK.

PART I. Because $OC = OB^a$, $AO + OC = AO + OB^b = AB^c$. But $AO + OC < AC^d$. Therefore $AB < AC^e$.

PART II. Because $OC > AC + AO^d$, OL (or $AL + AO$) $> AC + AO^e$. Whence, (by subtracting AO common) $AL > AC^f$.—(80).

PART III. If you say that AC , AD , are unequal, let AC be the greater; take $AF = AD$, and join OF . So must $OF = OD^g$; which is impossible^a. Therefore $AC = AD^h$.

PART IV. Because $OD = OE^a$, AO common, and angle $AOD < AOE^c$, AD (and consequently AC^i) $< AE^k$.

^a Def. 1. 3. ^b Ax. 3. ^c Ax. 1. ^d 9. 1. ^e Ax. 5. ^f Ax. 4. ^g 1. 1. ^h Ax. 11. ⁱ Part 3. ^k 13. 1.

PART V. $AG + GO > AH + HO$ ^a. Whence (by subtracting the equals GO, HO) $AG > AH$ ^b.

PART VI. If you say that AH, AI , are unequal, let AH be the greater; take $AM = AI$, and join OM . So must $OM = OI$ ^c, which is impossible ^d. Therefore $AH = AI$ ^e.

PART VII. $AI + IO > AK + KO$ ^f. Whence (by subtracting the equals IO, KO) AI (and consequently AH ^g) $> AK$ ^b.

COR. 1. THREE equal lines cannot be drawn to the circumference of a circle from any point which is not the center: consequently, if from any point in a circle three equal lines can be drawn to the circumference, that point is the center of the circle.

COR. 2. The circumferences of two circles cannot cut one another in more than two points; for if you say they will cut one another in three points, three equal lines may be drawn from one and the same point to the circumference of each circle ^d, which is therefore the center of both ^h; and their radii being thus equal, the circles must coincide ⁱ, contrary to hypothesis.

^a 9. I. ^b Ax. 4. ^c I. I. ^d Def. I. 3. ^e Ax. II.

^f 10. I. ^g Part 6. ^h Cor. I. II. ⁱ I. 3.

THEOREM XII.

IF, in a circle, two chords (AB, CD) cut one another, the rectangle contained under the parts (AE, BE) of the one, is equal to the rectangle contained under the parts (CE, DE) of the other.—(83, 84, 85, 86).

CASE 1. When both chords pass through the center (83).—Rect. AE, BE, = rect. CE, DE^a.

CASE 2. When but one chord (AB) passes thro' the center O (84, 85).—1. If AB is perpendicular to CD (84) draw OC. Then (because OC=OA) $OC+OE=AE^b$, and (because OC=OB) $OC-OE=BE^b$. Therefore, rect. AE, BE, = $OCsq.^c$ — $OEsq.^c$ = $CEsq.^d$ = rect. CE, DE^e.—2. If AB is not perpendicular to CD (85) let OM be perpendicular to CD, and draw OC. Then, as before, $OC+OE=AE$, and $OC-OE=BE$. Also $CM+EM=CE$, and (because $CM=DM^e$) $CM-EM=DE^b$. Therefore, rect. AE, BE, = rect. CE, DE^f.

CASE 3. When neither chord passes through the center (86).—Let MEN be the diameter. Then, rect. AE, BE, = rect. ME, NE^d, = rect. CE, DE^e.

^a Cor. 4. 1. 2. ^b Ax. 3. ^c Cor. 3. 6. 2. ^d Cor. 1. 7. 2.
^e Cor. 1. 4. 3. ^f Cor. 2. 7. 2. ^g Case 2.

T H E O R E M XIII.

IF, from any point (A) without a circle, two lines (AB, AC) be drawn, cutting the circle and terminating in the circumference; the rectangle contained under the whole (AB) and external part (AD) of the one, is equal to the rectangle contained under the whole (AC) and external part (AE) of the other.—(87, 88).

CASE 1. WHEN one of the lines (AB) passes through the center O (87).—Let OG be perpendicular to CE, and draw OC.—Then (because $AB=AO+OC^a$, $AD=AO-OC^a$, $AC=AG+GC^b$, and $AE=AG-GC^c$) $\text{rect. } AB, AD, = \text{rect. } AC, AE^d$.

CASE 2. When neither of the lines passes thro' the center (88).—Let AM pass through the center.—Then $\text{rect. } AB, AD, = \text{rect. } AM, AF^e, = \text{rect. } AC, AE^e$.

COR. 1. THE square of a tangent (AH) drawn from any point (A) without a circle, is equal to a rectangle contained under the whole of any line

E

^a Def. 1. 3. and Ax. 3.^b Ax. 1.^c Cor. 1. 4. 3. andAx. 3. ^d Cor. 2. 7. 2.^e Case 1.

(AB) drawn from the same point to the opposite part of the circumference and the external part (AD)—(88). For, AM, OH being drawn, $AHsq. = AOfsq. - OHsq. = \text{rect. AM, AF}^b$ (because $AM = AO + OH^c$ and $AF = AO - OH^c$) = rect. AB, AD^d .

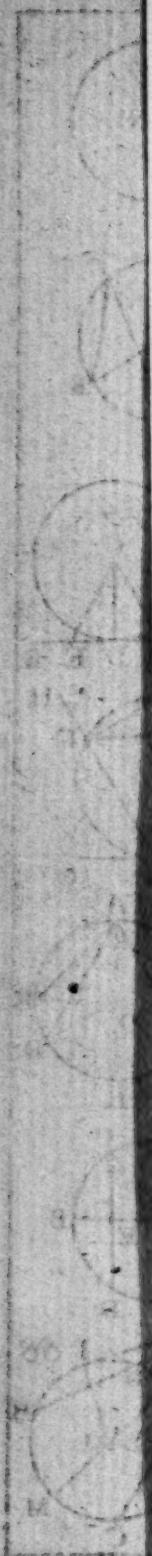
COR. 2. Two tangents (AH, AI) drawn from the same point (A) to a circle, are equal; because the square of each is equal to the same rectangle, AB, AD^e.

^a 6. 3. and Cor. 1. 7. 2. ^b Cor. 3. 6. 2. ^c Ax. 3.

^d Case 1. ^e Cor. 1. 13. 3.

SEC.





SECTION IV.

Of PROPORTION.

DEFINITIONS.

1. **F**OUR quantities (A, B, C, D) are said to be *proportional directly*, when the first has the same magnitude in comparison of the second, which the third has in comparison of the fourth. In this case, the first is said to have the same *ratio* to the second, as the third to the fourth; or more briefly, the first is said to be to the second, as the third to the fourth. This is commonly expressed by points; thus, $A : B :: C : D$.—*N. B.* The first and third quantities are called *antecedents*; and the second and fourth, *consequents*. Note also, If four quantities are said to be proportional, the order here mentioned is always to be understood, unless it be otherwise expressed.
2. Four quantities (A, B, C, D) are said to be proportional *alternately*, when the first, is to the third, as the second to the fourth; *inversely*, when the second is to the first as the fourth to the third; *conjunctly* or compoundedly, when the sum of the first and second is to the

first or second, as the sum of the third and fourth is to the third or fourth; *disjunctly* or *dividedly*, when the difference of the first and second is to the first or second, as the difference of the third and fourth is to the third or fourth; *mixedly*, when the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference; and *reciprocally*, when the first is to the third, as the fourth to the second. — Alternately, $A : C :: B : D$. — Inversely, $B : A :: D : C$. — Conjunctly, $A + B : A$ or $B :: C + D : C$ or D . — Disjunctly, $A \oslash B : A$ or $B :: C \oslash D : C$ or D . — Mixedly, $A + B : A \oslash B :: C + D : C \oslash D$. — Reciprocally, $A : C :: D : B$.

3. Any number of quantities ($A, B, C, D, \&c.$) are said to be *continually* proportional when the first is to the second, as the second to the third, as the third to the fourth, &c. Continual proportion is denoted thus $A : B : C : D, \&c.$
4. When three quantities are continually proportional, the first is said to have to the third a ratio *duplicate* of the ratio of the first to the second: and, when four quantities are continually proportional, the first is said to have to the fourth a ratio *triplicate* of the ratio of the first to the second.
5. When four quantities are proportional *directly*, the first and fourth are called the *extremes*; the other two, the *means*; and the fourth, a *fourth-proportional* to the other three.

6. When three quantities are *continually* proportional, the first and third are called the *extremes*; the middle quantity, the *mean*; and the third, a *third-proportional* to the other two.
7. Two sets of quantities, equal in number (A, B, C, D; P, Q, R, S,) are said to be *ordinately* proportional, when the first is to the second of the one set, as the first to the second of the other; and the second to the third of the one, as the second to the third of the other, &c. That is, when $A : B :: P : Q$, and $B : C :: Q : R$, &c.
8. Two sets of quantities equal in number (A, B, C, D; P, Q, R, S,) are said to be *inordinately* proportional, when the first is to the second of the one set, as the last save one to the last of the other; and the second to the third of the one, as the last save two to the last save one of the other, &c. That is, when $A : B :: R : S$ and $B : C :: Q : R$, &c.
9. A *multiple* of any quantity is twice, or thrice, &c. that quantity; and an *aliquot* part of any quantity, is one half, or one third, &c. of that quantity.
10. *Equimultiples* and *like aliquot parts* of two or more quantities, are such as are produced by the same number. Thus, 2 A, 2 B, are equimultiples, and $\frac{1}{2}A$, $\frac{1}{2}B$, like aliquot parts, of A, B.

A X I O M S.

1. **A**NY two quantities of the same kind may be supposed to consist of parts which are all equal to one another. For whatever number of equal parts one quantity is supposed to consist of, 'tis manifest the other will contain a certain number of such parts, either exactly, or with an overplus less than one of those parts. But the equal parts in the former quantity, may be conceived to be indefinitely small; in which case, the overplus of such parts in the latter must be nothing or indefinitely small.—In the same manner, it is manifest, that three or more quantities of the same kind may be supposed to consist of parts which are all equal to one another.—Hence it appears also, that any quantity may be considered as some multiple, or aliquot part, or multiple of an aliquot part, of any other quantity of the same kind.
2. If any two quantities consist of parts which are all equal to one another, either quantity is to the other as the number of those parts in the one is to the number of them in the other.—Thus, if A and B consist respectively of 2 and 3 parts all equal to one another, then $A : B :: 2 : 3$, and $B : A :: 3 : 2$.

3. If four quantities be proportional, whatever numbers of parts all equal to one another the first and second consist of, the same number of parts all equal to one another must the third and fourth consist of.—Thus, if $A : B :: C : D$, and if A and B consist respectively of 2 and 3 parts all equal to one another, C and D must also consist respectively of 2 and 3 parts all equal to one another; or (which is the same thing) if $A = \frac{2}{3}B$, C must $= \frac{2}{3}D$.
4. Whatever numbers of parts all equal to one another any two quantities consist of, the sum and difference of those quantities will consist of the sum and difference of those numbers of such parts.—If A consist of 5 parts and B of 3 parts all equal to one another, $A+B$ will consist of 8 such parts, and $A-B$ of 2 such parts.
5. If four quantities be proportional, the first is the same multiple, aliquot part, or multiple of an aliquot part, of the second, as the third is of the fourth.—If $A : B :: C : D$, and, if A be thrice, one third, or two thirds of B , C is also thrice, one third, or two thirds of D .
6. Ratios equal to the same or to equal ratios are equal.—If $A : B :: P : Q$ and $C : D :: P : Q$, $A : B :: C : D$; or, if $A : B :: P : Q$ and $C : D :: R : S$ and $P : Q :: R : S$, then $A : B :: C : D$.
7. Equals have to equals the same ratio.—If $A = B$ and $C = D$, then $A : C :: B : D$.

8. Unequal quantities have to equals unequal ratios, and the greatest quantity has the greatest ratio.—If $A < B$ and $C = D$, A is greater in comparison of C than B in comparison of D .
9. Quantities which have to equals unequal ratios are themselves unequal, and that is the greatest which has the greatest ratio. If $C = D$ and A have a greater ratio to C than B to D , then $A < B$.
10. If four quantities be proportional, and the first be equal to, greater, or less than the second, the third will accordingly be equal to, greater, or less than the fourth.—If $A : B :: C : D$; then, if $A = B$, $C = D$, if $A < B$, $C < D$, and if $A > B$, $C > D$.

POSTULATE.

THAT, to any three quantities, a fourth proportional may be assumed.

THEOREM

T H E O R E M I.

IF two antecedents be respectively to one consequent, as two other antecedents to another consequent; the two former antecedents are to one another as the two latter; and the sum or difference of the two former antecedents is to the former consequent, as the sum or difference of the two latter antecedents is to the latter consequent.—If $A : X :: C : Z$, and $B : X :: D : Z$; then, $A : B :: C : D$, and $A+B$ or $A \oslash B : X :: C+D$ or $C \oslash D : Z$.

LET A, B, X , consist respectively of p, q, r , parts all equal to one another^a; so will C, D, Z , consist respectively of p, q, r , parts all equal to one another^b; also $A+B, A \oslash B$, will consist respectively of $p+q, p \oslash q$, of the former equal parts, and $C+D, C \oslash D$ respectively of $p+q, p \oslash q$, of the latter^c.—Hence,

PART I. $A : B :: p : q$, and $C : D :: p : q$ ^d; therefore $A : B :: C : D$ ^e.

PART II. $A+B : X :: p+q : r$, and $C+D : Z :: p+q : r$ ^d; therefore $A+B : X :: C+D : Z$ ^e.

^a Ax. 1. 4. ^b Ax. 3. 4. ^c Ax. 4. 4. ^d Ax. 2. 4. ^e Ax. 6. 4.

PART III. $A \propto B : X :: p \propto q : r$, and $C \propto D : Z :: p \propto q : r^a$; therefore $A \propto B : X :: C \propto D : Z^b$.

COR. If any number of antecedents (A, B, C, &c.) be respectively to one consequent (X) as the same number of other antecedents (P, Q, R, &c.) to another consequent (Z), the sum of all the former antecedents is to the former consequent, as the sum of all the latter antecedents to the latter consequent.—For, $A+B : X :: P+Q : Z^c$, whence again (because $C : X :: R : Z$) $A+B+C : X :: P+Q+R : Z^c$, &c.

THEOREM II.

IF any number of quantities of the same kind be proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.—If $A : B :: C : D :: E : F$, &c. $A : B :: A+C+E \text{ \&c.} : B+D+F \text{ \&c.}$

LET A, B, consist respectively of p, q, parts, all equal to one another^d; so will C, D, consist respectively of p, q, parts, all equal to one another^e. Let X denote one of the equal parts in A, B, and Z one of the equal parts in C, D. Then $A+C=X+Z$ repeated p times; and $B+D$

^a AX. 2. 4. ^b AX. 6. 4. ^c I. 4. ^d AX. 1. 4. ^e AX. 3. 4.

$=X+Z$ repeated q times ^a. Therefore $A+B$, $B+D$, consist also respectively of p , q , parts, all equal to one another. Consequently, $A : B :: p : q :: A+C : B+D$ ^b.—Hence, $E : F :: A+C : B+D$ ^c. Consequently (by the preceding part) $E : F :: A+C+E : B+D+F$; and so on.

COR. 1. ANY two quantities (P , Q) of the same kind are to one another as their equimultiples (R , S). For, let $R=P+P+P$ &c. and $S=Q+Q+Q$ &c. then (since $P : Q :: P : Q :: P : Q$ &c. ^d) $P : Q :: P+P+P$ &c. : $Q+Q+Q$ &c. ^e; that is, $P : Q :: R : S$.

COR. 2. Any two quantities of the same kind are to one another as their like aliquot parts, those being equimultiples of these: whence also,

COR. 3. Any two quantities of the same kind are to one another as equimultiples of their like aliquot parts, by Cor. 1.

T H E O R E M III.

IF four quantities be proportional directly ($A : B :: C : D$) they are also proportional alternately, inversely, conjunctly, disjunctly, and mixedly.

^a Ax. 1. 1. ^b Ax. 2. 4. ^c Ax. 6. 4. ^d Def. 1. 4. ^e 2. 4.

PART I. BECAUSE $A : B :: C : D$; A , C , are equimultiples, or like aliquot parts, or equimultiples of like aliquot parts, of B , D^a . Therefore $A : C :: B : D^b$ *.

PART II. $B : B :: D : D^c$ and $A : B :: C : D^d$; therefore $B : A :: D : C^e$.

PART III. $A : A :: C : C^e$, and $B : A :: D : C^f$; therefore $A+B : A :: C+D : C^e$. Also $A : B :: C : D^d$, and $B : B :: D : D^e$; therefore $A+B : B :: C+D : D^e$.

PART IV. In the same manner as in Part 3. we derive $A \oslash B : A$ or $B :: C \oslash D : C$ or D .

PART V. $A+B : A :: C+D : C^g$, and $A \oslash B : A :: C \oslash D : C^h$; therefore $A+B : A \oslash B :: C+D : C \oslash D^e$.

COR. I. IF four quantities of the same kind be proportional, and the first be equal to, greater, or less than the third, the second will accordingly be equal to, greater, or less than the fourth; and, if the second be equal to, greater, or less than the fourth, the first will accordingly be equal to, greater, or less than the third.—For, if $A : B :: C : D$, then $A : C :: B : D^i$; whence it is manifest by Ax. 10,

^a Ax^s. 1. and 5. 4. ^b Cor^s. to 2. 4. ^c Def. 1. 4. ^d Hyp.

^e 1. 4. ^f Part 2. ^g Part 3. ^h Part 4. ⁱ Part 1.

* This ordering of the quantities, evidently requires that they be all of the same kind: the other variations do not.

COR. 2. IF the first of four proportionals of the same kind be the greatest, the fourth will be the least; if the second be the greatest, the third will be the least; if the third be the greatest, the second will be the least; and, if the fourth be the greatest, the first will be the least.—If $A : B :: C : D$, and A be the greatest, then, because $A < C^a$, $B < D^b$; and, because $A < B^a$, $C < D^c$. The rest follows from this, because $B : A :: D : C$, $C : A :: D : B$, and $D : C :: B : A^d$.—Hence, again,

COR. 3. The sum of the greatest and least of four proportionals of the same kind, exceeds the sum of the other two.—Let $A : B :: C : D$, and let A be the greatest; then D is the least^e. Now $A - B : A :: C - D : C^f$, and, consequently, $A : C :: A - B : C - D^g$. But $A < C^a$. Therefore $A - B < C - D^c$. To each of these unequals add $B + D$, and you have $A + D < B + C^h$.

T H E O R E M IV.

IF four quantities be proportional, then, instead of the first and second, or the third and fourth, their equimultiples, or like aliquot parts, or equimultiples of their like aliquot parts, may be substituted, and the

^a Hyp. ^b Cor. 1. 3. 4. ^c Ax. 10. 4. ^d 3. 4. ^e Cor. 2. 3. 4.
^f Part 4. ^g Part 1. ^h Ax. 4. 1.

quantities still remain proportional. Also, a similar substitution may be made instead of the first and third, or the second and fourth.—If $A : B :: C : D$, and P, Q , denote equimultiples, &c. of A, B , and R, S , equimultiples, &c. of C, D ; then, $P : Q :: C : D$, $A : B :: R : S$, and $P : Q :: R : S$.—Also, if $A : B :: C : D$, and P, Q , denote equimultiples, &c. of A, C , and R, S , equimultiples, &c. of B, D ; then $P : B :: Q : D$, $A : R :: C : S$, and $P : R :: Q : S$.

PART I. $P : Q :: A : B^a$, and $C : D :: A : B^b$; therefore $P : Q :: C : D^c$. $A : B :: C : D^b$, and $R : S :: C : D^a$; therefore $A : B :: R : S^c$. $P : Q :: A : B^a$, and $R : S :: C : D^a$; therefore $P : Q :: R : S^c$.

PART II. $P : A :: Q : C^d$, and $B : A :: D : C^e$; therefore $P : B :: Q : D^f$. $A : B :: C : D^b$, and $R : B :: S : D^d$; therefore $A : R :: C : S^f$. $P : B :: Q : D^g$, and $R : B :: S : D^d$; therefore $P : R :: Q : S^f$.

^a Cor^s. to 2. 4.^b Hyp.^c Ax. 6. 4.^d Def. I. 4.^e 3. 4.^f 1. 4.^g Part 2.

T H E O R E M V.

IF two sets of quantities be ordinately proportional, the first is to the last of the one set, as the first to the last of the other.—If A, B, C, D , and P, Q, R, S , be such that $A : B :: P : Q$, $B : C :: Q : R$, and $C : D :: R : S$; then $A : D :: P : S$.

BECAUSE $A : B :: P : Q^a$ and $C : B :: R : Q^b$, $A : C :: P : R^c$; and because $A : C :: P : R$ and $D : C :: S : R^b$, $A : D :: P : S^c$; and so on, to any number of quantities.

COR. 1. IF two sets of quantities be ordinately proportional, any two quantities in the one set, are to one another as the two corresponding quantities in the other^d: whence also (when the quantities in both sets are of the same kind) any quantity in the one set is to the corresponding quantity in the other, as any other quantity in the one is to its corresponding quantity in the other^b.

COR. 2. If two sets of quantities be ordinately proportional, the sum of the one set is to any quantity in it, as the sum of the other set is to the corresponding quantity in it. For, let the sets be

F 2

^a Hyp. ^b 3. 4. ^c 1. 4. ^d 5. 4.

denoted as above, and let X, Z , denote any two corresponding quantities; then $A : X :: P : Z$, $B : X :: Q : Z$, $C : X :: R : Z$, and $D : X :: S : Z^a$; consequently $A+B+C+D : X :: P+Q+R+S : Z^b$.—Hence also (when the quantities in both sets are of the same kind) the sum of the one set is to the sum of the other as any quantity in the one is to the corresponding quantity in the other^c.

THEOREM VI.

IF two sets of quantities be inordinately proportional, the first is to the last of the one set, as the first to the last of the other.—If A, B, C, D , and P, Q, R, S , be such that $A : B :: R : S$, $B : C :: Q : R$, and $C : D :: P : Q$, then $A : D :: P : S$.

LET M denote a fourth proportional to B, C, S , or let $B : C :: S : M^d$: Then (A, B, C , and R, S, M , being ordinally proportional) $A : C :: R : M^e$. But, (because $B : C :: Q : R$ and $B : C :: S : M^f$) $Q : R :: S : M^g$, and consequently, $Q : S :: R : M^b$. But $A : C :: R : M$. Therefore $A : C :: Q : S^g$.—Hence (A, C, D , and P, Q, S , being inordinately proportional) $A : D :: P : S$, by the preceding part: and so on, to any number of quantities.

^a Cor. 1. 5. 4.

^b Cor. 1. 4.

^c 3. 4.

^d Post.

^e 5. 4.

^f Hyp. ^g Ax. 6. 4.

S E C T I O N V.

Of PROPORTION in PLANE SURFACES.

D E F I N I T I O N.

SIMILAR straight-lined figures are those which have the angles of the one equal respectively to the angles of the other, and the sides opposite or about the equal angles proportional. Thus, if angle $A=D$, $B=E$, $C=F$, and $AB : DE :: BC : EF :: AC :: DF$, the triangles ABC , DEF are similar.—(101, 102).

T H E O R E M I.

PARALLELOGRAMS (AD , EH) having equal altitudes, are to one another as their bases (89, 90).—And parallelograms (TS , XW) having equal bases, are to one another as their altitudes (91, 92).

PART I. LET the bases CD, GH, consist respectively of P (3) Q (2) parts, all equal to one another^a; so will lines drawn from the divisions to AB, EF, and parallel to AC, EG, divide also AD, EH, respectively into P (3) Q (2) parts, all equal to one another^b. Therefore $AD : EH :: P : Q :: CD : GH^c$.

PART II. LET TA, VB, XC, YD, be perpendicular to TV, XY, and meet RS, UW produced, in A, B, C, D (91, 92).—Then $TS = VA$ and $XW = YC^d$. But $VA : YC :: TA : XC^e$. Therefore $TS : XW :: TA : XC^f$.

COR. 1. TRIANGLES having equal altitudes, are to one another as their bases; and, having equal bases, are to one another as their altitudes: because it holds true of their doubles.

COR. 2. Equal parallelograms, or equal triangles, having equal altitudes, have equal bases; and having equal bases, have equal altitudes. Therefore, also, equal parallelograms, (AD, EH) or equal triangles (CBD, GFH) having equal bases in the same line, and being on the same side of it, are between the same parallels (AF, CH). —(93).

^a Ax. 3. 4. ^b 2. 2. ^c Ax. 2. 4. ^d Cor. 2. 2. 2.

^e Part I. ^f Ax. 7. 4.

THEOREM

THEOREM II.

PARALLELOGRAMS (AD, FD) having one angle (BDC) in the one equal to one angle (EDG) in the other, and the sides (CD, BD; ED, GD) containing those angles reciprocally proportional, are equal.—And equal parallelograms (AD, FD) having one angle (BDC) in the one equal to one angle (EDG) in the other, have the sides (CD, BD; ED, GD) containing those angles reciprocally proportional.—(94).

LET the parallelograms be so placed, that BD, GD make one continued line; so will also CD, ED, make one continued line^a: let also AB, FE, be produced till they meet in H, forming a parallelogram (HD).

PART I. $AD : HD :: CD : ED$ and $FD : HD :: GD : BD$ ^b. But $CD : ED :: GD : BD$ ^c; therefore $AD : HD :: FD : HD$ ^d, and consequently $AD = FD$ ^e.

PART II. $AD : HD :: FD : HD$ ^f. But $AD : HD :: CD : ED$ and $FD : HD :: GD : BD$ ^b; therefore $CD : ED :: GD : BD$ ^d.

^a Cor. 6. 2. 1. ^b 1. 5. ^c Hyp. ^d Ax. 6. 4.

^e Cor. 1. 3. 4. ^f Ax. 7. 4.

COR. 1. TRIANGLES (BDC, EDG) having one angle in the one equal to one angle in the other; and the sides containing those angles reciprocally proportional, are equal: and equal triangles (BDC, EDG) having one angle in the one equal to one angle in the other, have the sides containing those angles reciprocally proportional: because they are halves of their respective parallelograms, AD, FD.

COR. 2. If four lines (AC, EG, GH, CD) be proportional (95, 96), the rectangle (AD) under the extremes is equal to the rectangle (EH) under the means; and if the rectangle (AD) under the extremes of four lines (AC, EG, GH, CD) be equal to the rectangle (EH) under the means, these lines are proportional.—For, in the first case, since angle $C = G^a$, and $AC : EG :: GH : CD^b$, $AD = EH^c$; and, in the second case, since $AD = EH$ and angle $C = G$, $AC : EG :: GH : CD^d$.—Hence,

COR. 3. If three lines be continually proportional, the rectangle under the extremes is equal to the square of the mean; and if the rectangle under the extremes of three lines be equal to the square of the mean, these lines are continually proportional.

^a Ax. 9. 1. ^b Hyp. ^c Part 1. ^d Part 2.

THEOREM

THEOREM III.

IF a line (AB) be parallel to any side (CE) of a triangle (CDE); it cuts the other sides (CD, ED) or those sides produced, proportionally.—And, if a line (AB) cut two sides (CD, ED) of any triangle (CDE) or those sides produced, proportionally; that line is parallel to the remaining side (CE).—(97, 98, 99).

DRAW AE, BC.

PART I. TRIANGLE $ABC = ABE^a$. Therefore $ABC : ADB :: ABE : ADB^b$. But $ABC : ADB :: AC : AD$ and $ABE : ADB :: BE : BD^c$. Therefore $AC : AD :: BE : BD^d$; whence also, $AD : BD :: AC : BE$, $AD : BD :: CD : ED$, &c.^e

PART II. Triangle $ABC : ADB :: AC : AD$ and $ABE : ADB :: BE : BD^c$. But $AC : AD :: BE : BD^f$. Therefore $ABC : ADB :: ABE : ADB^d$, and, consequently, $ABC = ABE^f$. Therefore AB is parallel to CE^h.

COR. If any number of parallel lines (AB, CD, EF) cut any number of other lines (GE,

^a Cor. 4. 2. 2. ^b Ax. 7. 4. ^c 1. 5. ^d Ax. 6. 4. ^e 3. 4.

^f Hyp. ^g Cor. 1. 3. 4. ^h Cor. 2. 1. 5.

GH, GI, GF) proceeding from a point (G) they cut them proportionally (100).—For, taking any two lines, EG, HG, we have $EG : HG :: EC : HL :: CG : LG :: CA : LK :: AG : KG^a$.

THEOREM IV.

IF two triangles (ABC, DEF) be mutually equiangular, or have three angles (A, B, C) of the one respectively equal to three angles (D, E, F) of the other; the sides (AB, AC, DE, DF) containing opposite equal angles (A, D) are proportional, provided those which are opposite equal angles be made the antecedents and consequents. — And if, of two triangles (ABC, DEF) the sides, containing all the angles be respectively proportional ($AB : AC :: DE : DF$; &c.) the triangles are mutually equiangular; and the equal angles are those which are opposite the antecedents or the consequents.—(101, 102).

PART I. IF $AB = DE$, $AC = DF^b$; and, consequently, $AB : AC :: DE : DF^c$.—If AB, DE be unequal, let AB be the greater; take $AG = DE$, and draw GH parallel to BC. Then (be-

^a Part I. ^b 5. 1. ^c Ax. 7. 4.

cause angle $A = D^a$, angle $AGH = B^b = E^a$, and $AG = DE^b$) $AH = DF^d$. Hence, $DE : DF :: AG : AH^c :: AB : AC^f$; whence also, $AB : DE :: AC : DF$.

PART II. If $AB = DE$, then (because $AB : AC :: DE : DF$ and $AB : BC :: DE : EF^a$) $AC = DF$ and $BC = EF^g$; and, consequently, angle $A = D$, $B = E$, and $C = F^h$.—If AB , DE be unequal, let AB be the greater; so will AC also exceed DF^g . Take $AG = DE$, $AH = DF$; and draw GH . Then (because $AB : AC :: DE : DF^a :: AG : AH^c$) GH is parallel to BC^f . Therefore, in triangles ABC , AGH (angle B being $= AGH$ and $C = AHG^b$) we have $AG : GH :: AB : BC^i :: DE : EF^a$. But $AG = DE^c$. Consequently $GH = EF^g$. Whence, angle $D = A$, $E = AGH = B$, and $F = AHG = C^h$.

COR. IF any number of lines (GE , GH , GI , GF) proceeding from a point (G) cut any number of parallel lines (AB , CD , EF) they cut them proportionally (100).—For, taking any two lines AB , CD (because triangles AGB , CGD , as also AGK , CGL , are mutually equiangular^b) we have $AB : AG :: CD : CG$ and $AG : AK :: CG : CL^i$; consequently $AB : AK :: CD : CL^k$.—In the same manner $AB : AM :: CD : CN$. Hence $AK : AB :: CL : CD$ and $AM : AB ::$

^a Hyp. ^b Cor. 2. 3. 1. ^c Cons. ^d Cor. 5. 1. ^e Ax. 7. 4.

^f 3. 5. ^g Cor. 1. 3. 4. ^h II. 1. ⁱ Part I. ^k 5. 4.

$CN : CD^a$; whence again, $KM : AB :: LN : CD^b$; and, consequently, $AB : KM :: CD : LN^a$.—Also, because $AB : AM :: CD : CN$, (disjunctly) $MB : AB :: ND : CD$, whence $AB : MB :: CD : ND^a$.—Therefore $AB : AK :: CD : CL$, $AB : KM :: CD : LN$, and $AB : MB :: CD : ND$; from which we derive, $AB : CD :: AK : CL :: KM : LN :: MB : ND^c$.

THEOREM V.

IF two triangles ($ABC, \bullet DEF$) have one angle (A) in the one, equal to one angle (D) in the other, and the sides (AB, AC, DE, DF) containing those angles proportional; the triangles are mutually equiangular, and the equal angles are those which are opposite the antecedents or the consequents.—(101, 102).

IF $AB = DE$, then (because $AB : AC :: DE : DF^f$) $AC = DF^d$; and, consequently, angle $B = E$, and $C = F^e$.—If AB, DE , be unequal, let AB be the greater; so will AC also exceed DF^d . Take $AG = DE$, $AH = DF$, and join GH . Then (because $AB : AC :: DE : DF^f :: AG : AH^g$) GH is parallel to BC^h ; consequently, angle $B = AGH^i = E^e$, and $C = AHG^i = F^e$.

^a 3. 4.^b 1. 4.^c Ax. 6. 4.^d Cor. 1. 3. 4.^e 1. 1.^f Hyp.^g Ax. 7. 4.^h 3. 5.ⁱ Cor. 2. 3. 1.

T H E O R E M VI.

IF two triangles (ABC, DEF) have one angle (A) in the one equal to one angle (D) in the other, and the sides (AB, BC, DE, EF) containing two other angles (B, E) proportional; the triangles are mutually equiangular, provided the remaining angles (C, F) be either both acute or both obtuse; and the equal angles are those which are opposite the antecedents or the consequents.—(101, 102).

IF $AB=DE$, then (because $AB : BC :: DE : EF^a$) $BC=EF^b$; and, consequently, angle $B=E$ and $C=F^c$.—If AB, DE , be unequal, let AB be the greater; take $AG=DE$, and draw GH parallel to BC . Then (angle AGH being $=B$ and $AHG=C^d$) we have $AG : GH :: AB : BC^e :: DE : EF^a$. But $AG=DE^f$. Consequently $GH=EF^b$. Whence (angle A being $=D^a$ and angles AHG, DFE , both acute or both obtuse, because angle $AHG=C^a$) we have angle $E=AGH=B$, and $F=AHG=C^c$.

COR. IF the hypotenuses and two other sides of two right-angled triangles be proportional, the triangles are mutually equiangular.

^a Hyp. ^b Cor. 1. 3. 4. ^c 12. 1. ^d Cor. 2. 3. 1. ^e 4. 5. ^f Conf.

THEOREM VII.

A Perpendicular (BD) drawn from the right angle (B) to the hypotenuse (AC) of a right-angled triangle (ABC) is a mean-proportional between the segments (AD, DC) of the hypotenuse; and each of the sides containing the right angle is a mean-proportional between the adjoining segment and the whole hypotenuse.— (104).

IN triangles ABC, ABD, angle $ABC = ADB^a$, A is common, and $C = ABD^b$. In triangles ABC, CBD, angle $ABC = CDB^a$, C is common, and $A = CBD^b$. Wherefore, also, in triangles ABD, CBD, angle $ABD = C$, $A = CBD$, and $ADB = CDB$. Hence, in triangles ABD, CBD, $AD : BD :: BD : CD$; in triangles ABC, ABD, $AC : AB :: AB : AD$; and, in triangles ABC, CBD, $AC : BC :: BC : CD^c$.

COR. A perpendicular (BD) drawn from any point in the circumference of a circle to the diameter, is a mean-proportional between the segments of the diameter; the square of the perpendicular is equal to a rectangle under the two segments of the diameter; and, if from that point

^a Ax. 9. I. ^b Cor. 5. 4. I. ^c 4. 5.

two chords be drawn to the extremities of the diameter, each chord will be a mean-proportional between the adjoining segment and the whole diameter; and the square of each chord will be equal to a rectangle under the whole diameter and its adjoining segment ^a.

T H E O R E M VIII.

IF two triangles (ABC, DEF) be mutually equiangular (angle $A=D$, $B=E$, and $C=F$) they are to one another as the squares (AH, DK) of any two corresponding sides.—(105, 106).

LET BM, EN, be perpendicular to AC, DF; and draw CG, FI.—Then, triangle ABC : ACG :: BM : AG ^b :: BM : AC ^c, and DEF : DFI :: EN : DI ^b :: EN : DF ^c. But AC : AB :: DF : DE ^d, and (because angle BAM = EDN ^e, AMB = DNE ^f, and ABM = DEN ^g) BM : AB :: EN : DE ^d. Therefore BM : AC :: EN : DF ^h; and, consequently, triangle ABC : ACG :: DEF : DFI ⁱ. Whence, ABC : DEF :: ACG : DFI ^k :: AH : DK ^l.

COR. THE squares of four proportional lines are themselves proportional.—For, if two equal

^a Cor. 5. 3. 3. and Cor. 3. 2. 5. ^b Cor. 1. 1. 5. ^c Ax. 7. 4.

^d 4. 5. ^e Hyp. ^f Ax. 9. 1. ^g Cor. 5. 4. 1. ^h 1. 4.

ⁱ Ax. 6. 4. ^k 3. 4. ^l Cor. 1. 2. 4.

angles, ABC, DEF , be formed (105, 106) and BA, BC, ED, EF , be taken respectively equal to the given proportionals; then, AC, DF being drawn, the triangles ABC, DEF will be mutually equiangular^a; and, therefore, we shall have $BA^2 : ED^2 :: \text{triangle } ABC : \text{triangle } DEF :: BC^2 : EF^2$ ^b; and, of consequence, $BA^2 : BC^2 :: ED^2 : EF^2$ ^c.

THEOREM IX.

SIMILAR straight-lined figures ($ABCDE, FGHIK$) having all the angles (A, B, C, D, E) of the one respectively equal to all the angles (F, G, H, I, K) of the other, and the sides about or opposite all the equal angles respectively proportional ($AB : AE :: FG : FK, \&c.$) are to one another as the squares of their corresponding sides (AB, FG).—(107, 108).

FROM any two equal angles, B, G , draw BE, BD, GK, GI , dividing the given figures into triangles.

THEN, because angle $A = F$ and $AB : AE :: FG : FK$, triangles ABE, FGK , are mutually equiangular^d.—Because angle $AED = FKI$ ^e, and $AEB = FKG$ ^d, angle $BED = GKI$ ^f; and, because $DE : EA :: IK : KF$ ^e, and $BE : EA :: GK$

^a 5. 5. ^b 8. 5. ^c 3. 4. ^d 5. 5. ^e Hyp. ^f Ax. 3. 1.

: KF^a , $DE : BE :: IK : GK^b$; therefore, triangles BED , GKI , are also mutually equiangular^a.—And, in a similar manner, it appears that triangles BDC , GIH , are mutually equiangular.—Hence, we have triangle $ABE : FGK (:: BEsq. : GKsq.) :: BED : GKI (:: BDsq. : GIsq.) :: BDC : GIH^c$. Therefore (by adding antecedents and consequents) $ABCDE : FGHIK :: ABE : FGK^d :: ABsq. : FGsq.^e$.

COR. 1. SIMILAR straight-lined figures are to one another as the squares of their corresponding diagonals (BE , GK)—For (as above) $ABCDE : FGHIK :: ABE : FGK :: BEsq. : GKsq.$

COR. 2. The sum of all the sides of any straight-lined figure, or its perimeter, is to the perimeter of a similar figure, as their corresponding sides or diagonals. For AB , BC , CD , DE , EA , and FG , GH , HI , IK , KF , being ordinately proportional^e, we have $AB+BC+CD+DE+EA : FG+GH+HI+IK+KF :: AB : FG^f :: BE : GK^a$.

COR. 3. If, on the first and second, and also on the third and fourth, of four proportional lines (AB , CD , EF , GH) similar figures be constructed (the given proportionals being made corresponding sides) these figures are likewise proportional (109, 110, 111, 112).—For $ABI : CDK :: ABsq. : CDsq.^h :: EFsq. : GHsq.^i :: EM : GO^h$.

^a 5. 5. ^b 1. 4. ^c 8. 5. ^d 2. 4. ^e Hyp. ^f Cor. 2. 5. 4.
^g 3. 4. ^h 9. 5. ⁱ Cor. 8. 5.

COR. 4. Similar straight-lined figures (ABI, CDK) are to one another in the duplicate ratio of their corresponding sides (AB, CD).—For, let $AB : CD :: CD : PQ^a$; then, $ABI : CDK :: ABsq. : CDsq.^b :: ABsq. : rect. AB, PQ^c, :: AB : PQ^d$.—(109, 110).

COR. 5. If similar straight-lined figures (AE, AG, CH) be constructed upon the three sides of any right-angled triangle (ABC) making those three sides corresponding sides of the figures, the figure upon the hypotenuse (AC) will be equal to the sum of the other two (113).—For $ABsq. : ACsq. :: AG : AE^b$, and $BCsq. : ACsq. :: CH : AE^b$. Therefore $ABsq. + BCsq. : ACsq. :: AG + CH : AE^c$. But $ABsq. + BCsq. = ACsq.^f$. Consequently $AG + CH = AE^g$.

THEOREM X.

IN a circle, any two arches (AB, CD) are to one another as the angles they subtend at the center (AOB, COD), and as the sectors they form (AOBA, CODC).—Also, any arch (AB) is to the whole circumference, as the angle it subtends at the center (AOB) is to four right angles, and as the sector it forms (AOBA) is to the whole circle.—(114).

^a Post. to Sect. 4. ^b 9. 5. ^c Cor. . 5. and Ax. 7. 4.

^d 1. 5. ^e 1. 4. ^f 7. 2. ^g Ax. 10. 4.

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LET the arches AB, CD, and the whole circumference, consist respectively of P (2), Q (3), and R (9) parts, all equal to one another^a; so will lines drawn from the divisions to the center (O) divide the angles AOB, COD, and the whole angular space at O (or 4 right angles) respectively into P (2), Q (3), and R (9) parts, all equal to one another^b; and they will divide also the sectors AOBA, CODC, and the whole circle, respectively into P (2), Q (3), and R (9) parts, all equal to one another^b.—Hence we have $AB : CD :: P : Q$, $AOB : COD :: P : Q$, and $AOBA : CODC :: P : Q$; also $AB : \text{circumf.} :: P : R$, $AOB : 4 \text{ right angles} :: P : R$, and $AOBA : \text{circle} :: P : R$ ^c. Consequently, $AB : CD :: AOB : COD :: AOBA : CODC$, and $AB : \text{circumf.} :: AOB : 4 \text{ right angles} :: AOBA : \text{circle}$ ^d.

COR. 1. IN equal circles, any two arches are to one another as the angles they subtend at the center, and as the sectors they form.

COR. 2. Any two arches, in the same or equal circles, are to one another as the angles they subtend at the circumference.

^a Ax. 1. 4. ^b 2. 3. and Cor^s. 1. and 2. ^c Ax. 2. 4.

^d Ax. 6. 4.

THEOREM XI.

THE circumferences (ACEA, GILG) of any two circles, are to one another as their radii or diameters.—(115, 116).

SUPPOSE the circumferences divided each into any the same number of equal parts, as in A, B, &c. and G, H, &c. Join those points, and draw the radii AP, BP, &c. and GQ, HQ, &c.—Then, the triangles in both circles, being isosceles, and having the angles at the centers all equal^a, are mutually equiangular^b; and we have chord AB : AP :: chord GH : GQ, chord BC : BP or AP :: chord HI : HQ or GQ, &c.^c. Hence, sum of chords in circle ACE : AP :: sum of chords in circle GIL : GQ^d, and (alternately) sum of chords in circle ACE : sum of chords in circle GIL :: AP : GQ^e. But, if we suppose the circumferences divided each into an indefinitely great number of equal parts, 'tis plain the sum of the chords in each circle will be indefinitely nearly equal to the circumference. Hence it is sufficiently manifest, that circumference ACEA : GILG :: AP : GQ, and consequently also as the diameters^f.

^a 10. 5.^b Cor. 5. 4. 1., 6. 1., and Ax. 3. 1.^c 4. 5.^d Cor. 1. 4.^e 3. 4.^f Cor. 1. 2. 4.

COR. ANY two arches (AB, GH) subtending equal angles at the centers or at the circumference of any two circles, are to one another as the radii, diameters, or circumferences of the circles.—For, $AB : \text{circumf. ACEA} :: \text{angle APB or GQH} : 4 \text{ right-angles} :: GH : \text{circumf. GILG}^a$. Consequently, $AB : GH :: \text{circumf. ACEA} : \text{circumf. GILG}^b :: AP : GQ^c$.

THEOREM XII.

ANY two circles (ACE, GIL) are to one another as the squares of their radii, diameters, or circumferences.—(115, 116).

THE same supposition and construction being made as in Theor. XI. we have triangle $ABP : GHQ :: BCP : HIQ^d$ &c. whence, sum of triangles in circle ACE : sum of triangles in circle GIL :: $ABP : GHQ^e :: APsq. : GQsq.^f$. But, if we suppose the circumferences divided each into an indefinitely great number of equal parts, 'tis plain the sum of the triangles in each circle will be indefinitely nearly equal to the circle. Hence 'tis sufficiently manifest, that circle ACE : GIL :: $APsq. : GQsq.$ and consequently,

^a 10. 5. ^b 3. 4. ^c 11. 5. ^d 1. 1. and Ax. 7. 4. ^e 2. 4. ^f 8. 5.

also, as the squares of the diameters or circumferences^a.

COR. 1. ANY two sectors (ABP, GHQ) contained under equal angles at the centers of any two circles, are to one another as the squares of the radii, diameters, or circumferences of the circles.—For, sector ABP : circle ACE :: angle APB or GQH : 4 right angles :: sector GHQ : circle GIL^b. Consequently, sector ABP : sector GHQ :: circle ACE : circle GIL^c : APsq. : GQsq.^d.

COR. 2. A circle having the hypotenuse of a right-angled triangle for its radius or diameter, is equal to the sum of two circles having the other sides for radii or diameters. The demonstration is similar to that in Cor. 5. Theor. 9. 5.

THEOREM XIII.

ANY circle (ACE) is equal to half a rectangle under the radius and the circumference.—(115).

THE triangles ABP, BCP, &c. being constructed as before, and a perpendicular, PR, drawn to AB, we have (because the triangles are

^a Cor. 8. 5. ^b 10. 5. ^c 3. 4. ^d 12. 5.

equal in every respect ^a) $ABP + BCP \&c. = \frac{1}{2}$ rect. $AB, PR, + \frac{1}{2}$ rect. $BC, PR, \&c.$ ^b $= \frac{1}{2}$ rect. under PR and the sum of the chords $AB, BC, \&c.$ ^c. But (as has been observed before) if we suppose the circumference divided into an indefinitely great number of equal parts, 'tis plain the sum of the triangles will be indefinitely nearly equal to the circle, the sum of the chords will be indefinitely nearly equal to the circumference, and the perpendicular PR will be indefinitely nearly equal to the radius. Hence 'tis sufficiently manifest, that the area of the circle is equal to half a rectangle under the radius and circumference.

^a I. 1. ^b Cor. 3. 2. 2. ^c Cor. 1. 5. 2.

SEC.

SECTION VI.

Of the SECTIONS of PLANES.

DEFINITIONS.

1. **A** *LINE* is said to be *perpendicular* to a *plane*, when it is perpendicular to all lines that can be drawn to it in the plane, Thus the line AB is perpendicular to the plane PQ (117).
2. One *plane* is said to be *perpendicular* to another, when all the lines that can be drawn in the one plane perpendicular to the line of section are perpendicular to the other plane. Thus the plane CD is perpendicular to the plane EF (128).
3. *Parallel* planes are those, which, though produced infinitely every way, would never meet as EF, GH (125).
4. A *plane* is said to *pass by a line*, when the line every where touches the plane. Thus the plane AD passes by the line PQ (123).
5. A plane passing by two or more lines, is said to be *the plane of those lines*. Thus AD is the plane of the lines PQ, RS (123).

T H E O R E M I.

THREE straight lines (AB, AC, BC) meeting one another, are in the same plane.—(117).

CONCEIVE a plane, BDEC, passing by BC, to revolve round BC till it touch the point A: then, because AB, AC are straight lines, and their extreme points in the plane, they, as well as BC, are in that plane^a.

COR. HENCE two straight lines (AB, AC) meeting or cutting one another, are in the same plane.

T H E O R E M II.

IF two planes (AB, CD) cut one another, the line of section (EF) is a straight line.—(118).

BECAUSE the extreme points, E, F, of the line of section are in both planes, a straight line joining those points must be wholly in both planes^a. But no line except the line of section can be wholly in both planes^b. Therefore the line of section, EF, is a straight line.

^a Def. 10. 1. ^b Def. 3. 1.

THEOREM III.

IF to each of two lines (CD, EF) cutting one another in a plane (PQ) a third line (AB) be perpendicular at the point of intersection, it is perpendicular to that plane. —(116).

MAKE BC, BD, BE, BF, all equal; join AC, AD, AE, AF, CF, ED; through B, in the plane PQ, draw MN, meeting CF, ED, in M, N; and join AM, AN.—Then, 1. Triangles ABC, ABD, ABE, ABF, being right-angled at B^a, and having BC=BD=BE=BF^b and AB common, have AC, AD, AE, AF, all equal^c. 2. Triangles CBF, EBD, having angle CBF=EBD^d, BC=BD, and BE=BF^b, have also CF=ED and angle BCM=BDN^e. 3. Triangles CBM, DBN, having angle CBM=DBN^d, angle BCM=BDN^e, and BC=BD^b, have also BM=BN and CM=DN^f. 4. Triangles ACF, AED, having AC=AD, AF=AE, and CF=ED^e, have also angle ACM=ADN^g. 5. Triangles ACM, ADN, having angle ACM=ADN, AC=AD, and CM=DN^e, have also AM=AN^c. 6. Triangles ABM, ABN, having AM=AN, BM=BN^e, and AB common, have also angle ABM=ABN^g. Therefore AB

^a Hyp. ^b Conf. ^c I. I. ^d Cor. 5. 2. I. ^e Dem. of 3. 6.

^f Cor. 5. I. ^g II. I.

is perpendicular to MN^a ; and, in a similar manner it may be proved to be perpendicular to any other line drawn through B in the plane PQ .—Therefore AB is perpendicular to PQ^b .

T H E O R E M IV.

IF to each of three lines (CB, DB, EB) meeting or cutting one another in a point (B) another line (AB) be perpendicular, those three lines (CB, DB, EB) are in one and the same plane.—(119).

If you say that any one of the lines, EB , is above or below the plane, PQ , of the other two, CB, DB ; let BF be the common section of PQ and the plane of AB, BE . Then, angle $ABF =$ a right angle $^c =$ angle ABE^d ; which is impossible e . Therefore, EB is in the same plane with CB, DB^f .

COR. HENCE, if one line be perpendicular to any number of lines meeting together, those lines to which it is perpendicular are all in one and the same plane.

^a Def. 7. 1. ^b Def. 1. 6. ^c 3. 6. ^d Hyp. ^e Ax. 1. 1.
^f Ax. 11. 1.

THEOREM V.

IF two lines (AB, CD) be perpendicular to the same plane (PQ) they are parallel.—(120).

IN PQ draw $DE=AB$ and perpendicular to BD, and join AD, AE, and BE.—Then, in triangles ADB, EDB (because $AB=DE^a$, BD common, and angle $ABD=EDB^b$) $AD=BE^c$: whence, in triangles ADE, ABE, (because $AD=BE^d$, $DE=AB^a$, and AE common) angle $ADE=ABE^e=a$ right angle f . ED is therefore perpendicular to AD, as well as to BD and CD g . Consequently, CD being in the same plane with AD and BD h , is in the same plane also with AB i . But AB, CD, are both perpendicular to BD f . Therefore, AB, CD, are parallel k .

COR. 1. FROM the same point there can be drawn only one perpendicular to any plane; for all perpendiculars to the same plane are parallel l ; but two parallel lines cannot be drawn from the same point m .

COR. 2. If one plane (AD) be perpendicular to another plane (EH) a line perpendicular to the latter plane at any point (Q) in the line of section

^a Conf. ^b Conf. and Ax. 9. 1. ^c 1. 1. ^d Dem. of 5. 6.

^e 11. 1. ^f Hyp. and Def. 1. 6. ^g Conf. and Def. 1. 6.

^h 4. 6. ⁱ 1. 6. ^k Cor. 3. 3. ^l 5. 6. ^m Def. 19. 1.

(CD) must be in the former plane (122). For, if QP be drawn in AD perpendicular to CD , it is perpendicular to EH^a , and is the only line that can be drawn perpendicular to EH at the point Q^b .

T H E O R E M VI.

IF two lines (AB, CD) be parallel, and one of them (AB) be perpendicular to a plane (PQ) the other (CD) is also perpendicular to the same plane.—(120).

IN PQ draw $DE=AB$ and perpendicular to BD ; and join AD, AE , and BE .—Then (as in Theor. 5.) in triangles ADB, EDB (because $AB=DE^c$, BD common, and angle $ABD=EDB^d$) $AD=BE^e$: whence, in triangles ADE, ABE (because $AD=BE^e$, $DE=AB^c$ and AE common) angle $ADE=ABE^g=a$ right angle h . ED , therefore, being perpendicular to AD and BD , is perpendicular to the plane of those lines or of the given parallels, and consequently perpendicular to CD^i . But angle $ABD+CDB=2$ right angles k . Consequently (ABD being right l) CDB is a right angle m . Therefore, since CD is perpendicular to BD as well as to ED , it is perpendicular to PQ^i .

^a Def. 2. 6. ^b Cor. 1. 5. 6. ^c Conf. ^d Conf. and Ax. 9. 1.

^e 1. 1. ^f Dem. of 6. 6. ^g 11. 1. ^h Hyp. and Def. 1. 6.

ⁱ 3. 6. ^k Cor. 2. 3. 1. ^l Hyp. ^m Ax. 3. 1.

THEOREM VII.

IF two lines (AB, CD) be parallel to a third line (EF) they are parallel to one another, though they be not all in one and the same plane.—(121).

IN the plane of AB, EF, let MO be perpendicular to EF; and, in the plane of CD, EF, let NO be perpendicular to EF.—Then, because EF is perpendicular to MO, NO^a, it is perpendicular to the plane of those lines^b; and, because AB, CD, are parallel to EF, they are also perpendicular to the same plane^c, and consequently parallel to one another^d.

THEOREM VIII.

IF one plane (AD) pass by a line (PQ) which is perpendicular to another plane (EH), each of these planes is perpendicular to the other.—(122).

PART I. IN the plane AD, draw any line, RS, perpendicular to the line of section CD; so is it parallel to PQ^e, and, consequently, perpen-

^a Conf. ^b 3. 6. ^c 6. 6. ^d 5. 6. ^e Cor. 3. 3. 1.

dicular to the plane EH^a . In the same manner, any other line that can be drawn in AD perpendicular to the line of section, is perpendicular to EH. Therefore AD is perpendicular to EH^b .

PART II. In the plane EH, from the point Q, draw QT perpendicular to the line of section, CD. Then, because QT is also perpendicular to PQ^c , it is perpendicular to the plane AD^d , those lines CD, PQ, being in AD. Therefore the plane EH, passing by QT, is perpendicular to the plane AD^e .

COR. If one plane be perpendicular to another, the latter plane is also perpendicular to the former.

T H E O R E M IX.

PLANES (AB, CD) to which the same line (EF) is perpendicular, are parallel to one another.—(123).

If you say that AB, CD, are not parallel, let them meet; and let their line of section be GH, to any point in which, O, draw EO, FO.—Then, in triangle FOE, angles E, F, are both right angles f ; which is impossible g . Therefore AB is parallel to CD^h .

^a 6. 6. ^b Def. 2. 6. ^c Hyp. and Def. 2. 6. ^d 3. 6. ^e Part I.

^f Hyp. and Def. 1. 6. ^g Cor. 2, 4. 1. ^h Ax. 11. 1.

THEOREM X.

THE lines of section (AB, CD) made by one plane (AD) cutting two parallel planes (EF, GH) are parallel.—(124).

AB, CD, are in the same plane, AD^a; and being also in parallel planes, EF, GH^a, they can never meet^b. Therefore they are parallel^c.

COR. 1. PARALLEL lines (MN, PQ) terminated by two parallel planes (EF, GH) are equal; and the lines (MP, NQ) joining their extremes, are also equal and parallel. For MP, NQ (being lines of section in EF, GH, made by the plane of MN, PQ^a) are parallel^d; therefore also, MN = PQ and MP = NQ^e.

COR. 2. Hence, all lines drawn from one of two parallel planes perpendicular to the other plane, are equal; because those lines are parallel^f.

COR. 3. If a line (MN) be perpendicular to one of two parallel planes (GH) it is also perpendicular to the other (EF).—For, if from the point, M, where it meets EF, any line whatever, MP, be drawn in EF, and PQ be perpendicular

^a Hyp. ^b Def. 3. 6. ^c Def. 19. 1. ^d 10. 6. ^e Cor. 2. 1. 2. ^f 5. 6.

to GH; then (joining NQ) MP, NQ, are parallel^a. Therefore MN is perpendicular to MP^b; consequently, MN is perpendicular to EF^c, MP being any line whatever drawn in EF from the point M. The same is also manifest from Theor. 9.—Hence,

COR. 4. If a plane (AD) be perpendicular to one of two parallel planes (GH) it is also perpendicular to the other (EF).—For, draw MN in AD and perpendicular to CD, it is perpendicular to GH^d, and consequently perpendicular to EF^e. Therefore AD is perpendicular to EF^f.

T H E O R E M X I.

IF the legs (AB, AC) of one angle (BAC) formed in one plane, be respectively parallel to the legs (DE, DF) of another angle (EDF) formed in another plane; the angles are equal, and the planes in which they are formed are parallel.—(125, 126).

PART I. MAKE $AB=DE$, $AC=DF$, and draw AD, BE, CF, BC, and EF (125).—Then (because AB is equal and parallel to DE, and AC equal and parallel to DF) BE, CF, are each equal and

^a Cor. 1. 10. 6. ^b Cor. 4. 3. 1. ^c Def. 1. 6. ^d Def. 2. 6.

^e Cor. 3. 10. 6. ^f 8. 6.

parallel to AD^a , and, therefore, equal and parallel to one another^b. Therefore $BC=EF^a$; and, consequently (because also, $AB=DE$ and $AC=DF$) $BAC=EDF^c$.

PART II. Let AG be perpendicular to the plane of DE, DF ; and draw GH, GI , respectively parallel to DE, DF (126).—Then (GH, GI , being respectively parallel to AB, AC^b), AG , which is perpendicular to GH, GI , is also perpendicular to AB, AC^d , and, consequently, to the plane of AB, AC^e . Therefore (since AG is perpendicular to the plane of AB, AC , and to the plane of DE, EF) these planes are parallel to one another^f.

THEOREM XII.

IF two planes (AB, CD) cutting one another be both perpendicular to a third plane (EF), their line of section (GH) is perpendicular to the same plane (EF).—(127).

BECAUSE AB, CD , are both perpendicular to EF^g , a line perpendicular to EF at the point H , must be in both these planes^h, and therefore must be their line of section, GH .

^a 3. 2. ^b 7. 6. ^c II. I. ^d Cor. 4. 3. I. ^e 3. 6. ^f 9. 6.
^g Hyp. ^h Cor. 2. 5. 6.

T H E O R E M XIII.

LINES (AB, CD) cut by parallel planes (EF, GH, IK) are cut proportionally.—(129).

DRAW MR, cutting GH in S; and join MP, NS, QS, OR.—Then (in the plane of MO, MR) NS is parallel to OR^a; and (in the plane of MR, PR) QS is parallel to MP^a. Therefore, in triangles MOR, MRP, we have MN : NO :: MS : SR :: PQ : QR^b.

COR. ANY two lines (MO, MR) proceeding from a point, are cut proportionally by parallel planes. For MN : NO :: MS : SR^d.

T H E O R E M XIV.

IF from any two points (M, N) in a line (AB) cutting a plane (XZ) whether these points be the same side, or on opposite sides of the plane, two lines (MP, NQ) be drawn perpendicular to the plane; a line (PQ) passing through, the points (P, Q) in which these perpendiculars meet the

^a 10. 6. ^b 3. 5.

plane, will pass through the point, O, in which the first-mentioned line (AB) cuts the plane.—(129, 130).

BECAUSE MP, NQ, are perpendicular to XZ, they are parallel to one another and in one plane^a. Therefore AB, PQ, are in one plane^b. But, since AB cuts XZ, it cannot be parallel to any line, PQ, in XZ^c. Consequently AB, PQ, must intersect each other; and, since PQ is in XZ, 'tis manifest, their point of intersection must be that one (O) in which AB cuts XZ.

COR. 1. IF, in the plane XZ, from the points P, Q, two parallels, PR, QS, be drawn, such that $PR : QS :: MP : NQ$ (PR, QS, being on the same or different sides of PQ, according as MP, NQ, are on the same or different sides of XZ), then, a line passing through the points R, S, will also pass through the point, O, in which AB cuts the plane XZ. Draw RO, SO. Then (because of parallels) $OP : OQ :: MP : NQ^d :: PR : QS^e$, whence $OP : PR :: OQ : QS^f$. Therefore (angle OPR being equal OQS^g) angle POR = QOS^h; and, consequently, OR, OS, make one continued lineⁱ.

COR. 2. Because of parallels, we have also, $ON : NM :: OQ : QP :: OS : SR$, $OM : ON :: OP : OQ :: OR : OS$, &c.

^a 5. 6. and Def. 19. 1.

^b Def. 10. 1.

^c Def. 19. 1.

^e Hyp.

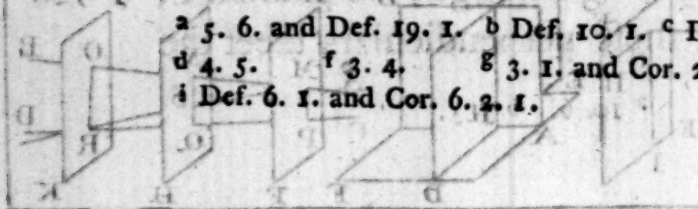
^d 4. 5.

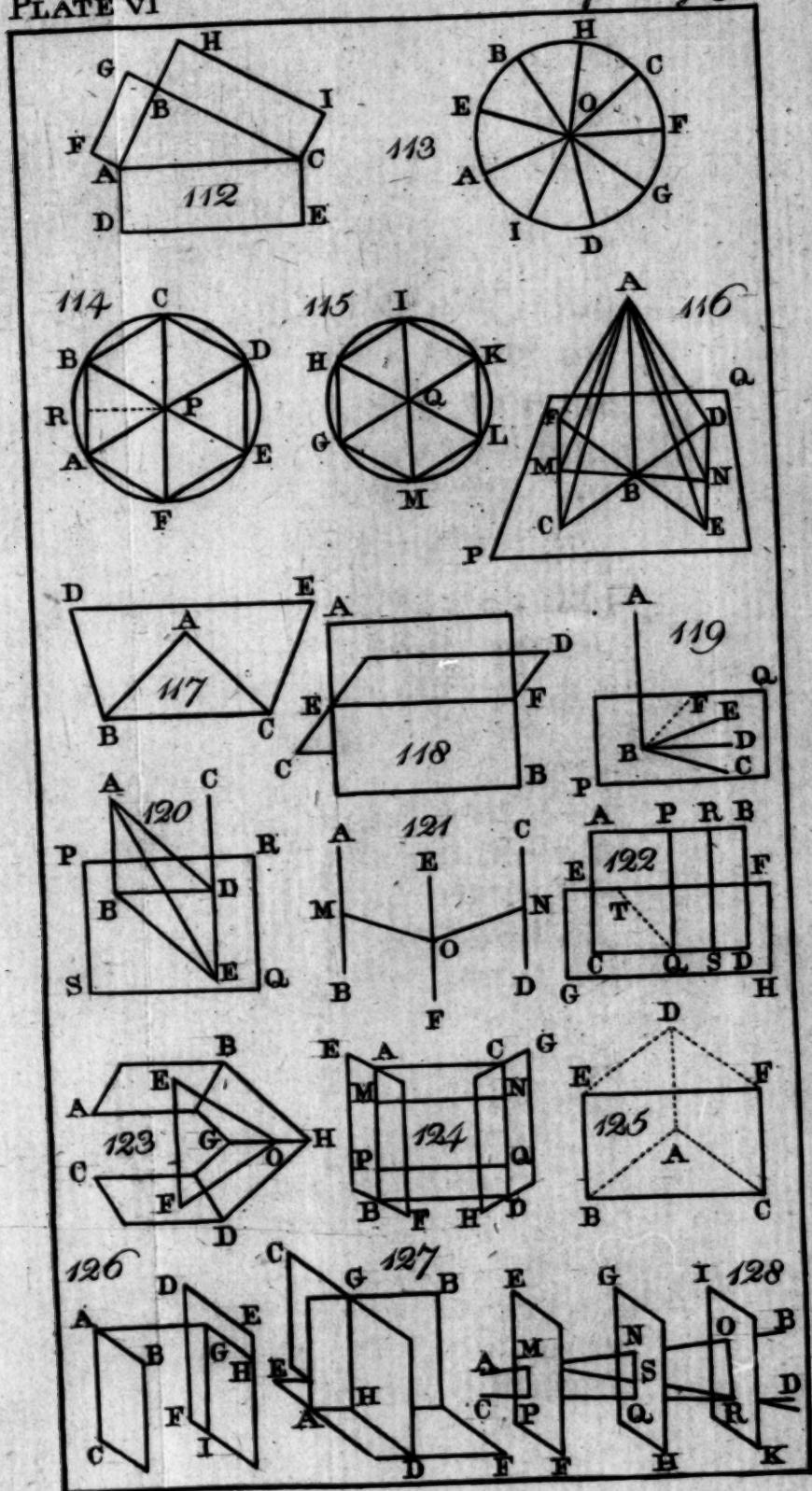
^f 3. 4.

^g 3. 1. and Cor. 2. 3. 1.

^h 5. 5.

ⁱ Def. 6. 1. and Cor. 6. 2. 1.





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THEOREM XV.

IF, from the angular points (A, B) of two equal angles (CAD, EBF) two lines (AG, BH) be drawn out of the planes of those angles, making with the legs of the same angles, two angles respectively equal to two ($GAC=HBE$, and $GAD=HBF$), those two lines (AG, BH) are to one another as two lines (GI, HK) drawn from their extreme points perpendicular to the planes of the first-mentioned angles.—(131, 132, 133).

CASE 1. WHEN AG, BH, are equal (131, 133).—In the planes GAC, GAD, HBE, HBF, draw GL, GM, HN, HO, perpendicular to AG, BH, and meeting AC, AD, BE, BF, in L, M, N, O; draw AI, BK, and LM, NO, meeting AI, BK (produced) in P, Q; and join GP, HQ.—Then, in triangles AGL, BHN (angle GAL being $=HBN^a$, angle $AGL=BHN^b$, and $AG=BH^a$) we have $AL=BN^c$. In a similar manner, in triangles AGM, BHO, we have $AM=BO$. Hence, in triangles ALM, BNO (angle LAM being $=NBO$) we have angle $ALP=BNQ^d$.—But (because AG is perpendicular to GL and GM) AG, and, consequently, the plane AGP passing by it, is perpendicular to the plane

^a Hyp. ^b Conf. and Ax. 9. 1. ^c Cor. 5. 1. ^d 1. 1.

LGM^a: and (because GI is perpendicular to the plane LAM) the plane AGP is also perpendicular to the plane LAM^b. Therefore, the planes LGM, LAM, being both perpendicular to the plane AGP^c, their line of section, LM, is perpendicular to the same plane^d, and, consequently, perpendicular to AP^e. In the same manner, NO is perpendicular to BQ. Therefore angle APL=BQN^f. But it has been proved that angle ALP=BNQ and AL=BN. Consequently, in triangles APL, BQN, AP=BQ^g. Now, AG, being perpendicular to both GL and GM, in the plane LGM, it is also perpendicular to GP^h; and, in like manner, BH is perpendicular to HQ. Therefore, in triangles AGP, BHQ, (AP being=BQ, AG=BH, and angles AGP, BHQ, right angles) we have angle GAP=HBQⁱ; whence, in triangles GAI, HBK, (angle GAI being=HBK and angle GIA=HKB^j) we have AG : BH :: GI : HK^k.

CASE 2. When AG, BH, are unequal (132, 133).—In the greater, AG, take AR=BH; draw AI, and also RS parallel to GI, and, consequently, perpendicular to the plane of CAD^l. Then (by Case 1.) AR : BH :: RS : HK. Therefore BH : HK :: AR : RS^m :: AG : GIⁿ; and, consequently, AG : BH :: GI : HK^m.

COR. HENCE the two perpendiculars (GI, HK) make equal angles with the lines (AG, BH) drawn from the angular points (A, B) and also subtend equal angles at those points.

^a 3. and 8. 7. ^b 8. 7. ^c Cor. 8. 7. ^d 12. 7. ^e Def. 1. 7.

^f Ax. 9. 1. ^g Cor. 5. 1. ^h 3. 7. ⁱ Cor. 12. 1. ^k 4. 5.

^l 6. 6. ^m 3. 4. ⁿ 3. 5.

SECTION VII.

Of SOLIDS.

DEFINITIONS.

1. **A** *PRISM* (AD, 134) is a solid which may be conceived to be formed by any straight-lined plane surface (CD) moving from the plane of that surface in straight lines (CA, GE, HF, DB, KI) and in a position always parallel to that plane.—Hence, the sides of a prism are parallelograms equal in number to the sides of the base; and the top, as well as every section (PQ) of a prism made by a plane parallel to the base, is equal and similar to the base (CD).
2. A *parallelepipedon* is a prism, of which the generating surface is a parallelogram; as AD (135).—Hence, the ends, as well as sides of a parallelepipedon are parallelograms.
3. A *cube* is a prism, of which the generating surface is a square, and the sides also squares; as AD (136).—Hence, the ends, as well as sides of a cube, are squares; and the sides are perpendicular to the ends.

4. A *cylinder* (AD, 137) is a solid which may be conceived to be formed by any circle (CD) moving from the plane of that circle in straight lines (CA, DB) and in a position always parallel to that plane.—Hence, the top, as well as every section of a cylinder parallel to the base, is a circle equal to the base. The *axis* of a cylinder is a straight line joining the centers of the top and base, as EF.
5. A *pyramid* (ABD, 138) is a solid which may be conceived to be described by the motion of a straight line (AB) round the sides of any straight-lined plane surface (BD) one extremity (A) of the line remaining fixed. — Hence, the sides of a pyramid are triangles equal in number to the sides of the base.
6. A *cone* (FGI, 139) is a solid which may be conceived to be described by the motion of a straight line (FG) round the circumference of a circle (GI) one extremity (F) of the line remaining fixed.—The *axis* of a cone is a straight line joining the vertex or top of the cone, and the center of the base, as FK.
7. A *sphere* or *globe* (AB, 140) is a solid which may be conceived to be formed by the revolution of a semi-circle (CBD) round its diameter (CD) which remains fixed.—The *center* of the sphere, O, is that of the circle; whence 'tis manifest, that all lines drawn from the center to the surface of a sphere (called *radii* of the sphere) as well as all lines passing through

the center and terminated both ways by the surface (called *diameters* of the sphere) are equal. 'Tis also manifest, that all sections of a sphere made by a plane are circles; and that any such section passing through the center, divides the sphere into two equal parts or hemispheres.

8. The altitude of a prism, cylinder, pyramid, or cone, is a perpendicular let fall from the top upon the plane of the base.

A X I O M.

IF any two solids (134, 135; or 138, 141) having the same altitude, have also, at all equal altitudes, their sections (PQ, RS; or PR, TU) made by planes parallel to the bases equal, the solids themselves are equal.

T H E O R E M I.

PRISMS and cylinders having equal bases and equal altitudes, are themselves equal.—(134, 135, 137).

THE proposed solids, having the same altitude^a; and having their sections at all equal altitudes parallel to the bases, also equal^b, are themselves equal^c.

^a Hyp. ^b Def^s. 1. and 4. of 7. ^c Ax. to 7.

THEOREM II.

PRISMS and cylinders, having equal altitudes, are to one another as their bases; and, having equal bases, are to one another as their altitudes.—(134, 135, 137).

PART I. LET any two bases consist respectively of M, N parts, all equal to one another^a: then, in the formation of the solids, 'tis manifest, these equal parts of the bases form in the given solids respectively, M, N solids or parts^b, which parts (if their altitudes be equal) are all equal to one another^c. Consequently, in this case, the given solids are to one another as their bases^d.

PART II. Let any two of the altitudes consist respectively of M, N , parts or altitudes, all equal to one another^e; then, if, by sections parallel to the bases, we conceive the given solids cut at those equal altitudes, the given solids will thus be divided respectively into M, N parts, which parts (if the bases be equal) must be all equal to one another^f. Consequently, in this case, the given solids are to one another as their altitudes^g.

^a Ax. I. 4. ^b Def. I. 7. ^c I. 7. ^d Ax^s. 2. and 6. of 4.

^e I. 7. ^f Ax^s. 2. and 6. of 4.

COR. 1. HENCE prisms, having equal altitudes and similar bases, are to one another as the squares of the corresponding sides or diagonals of their bases ^a : and cylinders, having equal altitudes, are to one another as the squares of the radii, diameters, or circumferences of their bases ^b.

COR. 2. Equal prisms or cylinders, having equal altitudes, have equal bases ; and, having equal bases, have equal altitudes.

T H E O R E M III.

PYRAMIDS, having equal bases and equal altitudes, are equal. — (138, 141).

LET PR, TU, be sections of the pyramids parallel to the bases, and at any equal altitudes, KM, LN.—Then, in the pyramid ABD, because PR is parallel to BD, PS is parallel to BC, PQ to BE &c. ^c consequently angle SPQ=CBE, PQR=BED, &c. ^d : and (because of parallels in triangles APS, ABC; APQ, ABE; AQR, AED; &c.) PS : BC (:: AP : AB) :: PQ : BE (:: AQ : AE) :: QR : ED ^e &c. Therefore PR, BD, are simi-

^a 9. 5. ^b 12. 5. ^c 10. 6. ^d 11. 6. ^e 4. 5.

lar. Hence (drawing PM, BK, which, being in the planes of PR, BD, must be perpendicular to AK^a, and consequently parallel to one another^b)
 $PR : BD :: PSsq. : BCsq. ^c :: APsq. : ABsq. : : AMsq. : AKsq. ^d$. In the same manner, in the pyramid FGI (drawing TN, GL) $TU : GI :: FNsq. : FLsq.$ But $AMsq. : AKsq. :: FNsq. : FLsq. ^e$. Therefore $PR : BD :: TU : GI ^f$. But $BD = GI ^g$. Therefore $PR = TU ^h$. Consequently $pyr. ABD = pyr. FGI ^i$.

COR. 1. ANY cone is equal to a pyramid of an equal base and altitude. For, if the base of the cone be conceived to consist of an indefinitely great number of equal parts, and a straight-lined surface be formed by joining the several divisions, 'tis manifest the motion of a line round that figure, while the other extremity remained fixed at the vertex of the cone, would describe a pyramid indefinitely nearly equal to the cone, and equal to any other pyramid of an equal base and altitude with itself^k; whence 'tis sufficiently manifest, that the cone must also be equal to any pyramid of an equal base and altitude with itself.

COR. 2. Any two cones, having equal bases and equal altitudes, are equal to one another; being each equal to a pyramid of the same base and altitude^l.

^a Cor. 3. 10. 6. and Def. 1. 6. ^b Cor. 3. 3. 1. ^c 9. 5.

^d Cor. 8. 5. ^e Ax. 7. 4. ^f Ax. 6. 4. ^g Hyp. ^h Cor. 1. 3. 4.

ⁱ Ax. 10. 7. ^k 3. 7. ^l Cor. 1. 3. 7.

COR. 3. From the demonstration of the Theorem and Cor. 1. it appears, that, in a pyramid, any section parallel to the base is similar to the base; and that, in a cone, every such section is a circle.

THEOREM IV.

ANY prism is equal to thrice a pyramid of the same base and altitude.

I. IF the prism have a triangular base, as CE (142), draw the diagonals AE, AF, and BF.—Then, a plane passing by AE and AF, will cut off a triangular pyramid, ADEF; and another plane passing by AF and BF, will divide the remainder of the prism into two other triangular pyramids, ACFB and ABEF. Now, the pyramids ACFB and ABEF, having equal bases, CFB, BEF^a, and the same altitude (their bases being in the same plane, and their vertices meeting in the same point, A) are equal to one another^b; and the pyramids ADEF, ACFB, having equal bases, DEF, ABC^c, and equal altitudes^d, are also equal to one another^b. Therefore, the prism CE consists of three triangular pyramids, ADEF, ACFB, and ABEF, which are equal to one another, and, consequently, is equal to thrice one of them, ADEF, or to any other triangular pyramid having the same or an equal base and altitude^b.

^a Cor. 1. I. 2. ^b 3. 7. ^c Def. 1. 7. ^d Cor. 2. 10. 6.

II. If the prism have not a triangular base, as GH (143) yet 'tis manifest that it consists of as many triangular prisms GO, IH, as the base contains triangles (MNO, ONH) formed by diagonals; and 'tis also manifest, that a pyramid PQR (144) of the same (or an equal and similar) base and altitude consists of the same number of triangular pyramids, PQST, PRST, having bases respectively equal to the bases of the corresponding prisms, and equal altitudes. Therefore each prism is (by the preceding part) thrice the corresponding pyramid; and, consequently, the whole prism GH is thrice the whole pyramid PQR.

COR. 1. HENCE, every prism or cylinder is equal to thrice a pyramid or cone of an equal base and altitude; every cylinder being equal to a prism of the same base and altitude^a, which prism is equal to thrice a pyramid, and, consequently, to thrice a cone, of the same base and altitude^b.

COR. 2. Hence also, pyramids or cones, having equal altitudes, are to one another as their bases; and, having equal bases, are to one another as their altitudes; because their equimultiples are in that ratio^c.

COR. 3. Hence, equal pyramids or cones, having equal altitudes, have equal bases; and, having equal bases, have equal altitudes.

COR. 4. Pyramids, having equal altitudes and similar bases, are to one another as the squares of

^a I. 7. ^b 4. 7. and Cor. 1. 3. 7. ^c 2. 7.

the corresponding sides or diagonals of their bases; and cones, having equal altitudes, are to one another as the squares of the radii, diameters, or circumferences, of their bases; because their bases are in that ratio ^a.

T H E O R E M V.

PRISMS and cylinders, as also pyramids and cones, having their bases and altitudes reciprocally proportional, are equal.

LET the prisms AKD, AHD (134, 136) have base GD : base CD :: alt. of AHD : alt. of AKD. Let AKD have the greater altitude; in which, let PKD be a prism, having the same altitude as AHD.—Then, PKD : AHD :: GD : CD ^b :: alt. of AHD : alt. of AKD ^c :: alt. of PKD : alt. of AKD ^d :: PKD : AKD ^b. Therefore AKD = AHD ^e.—In the same manner, it will appear, that any two cylinders, or any prism and cylinder, having their bases and altitudes reciprocally proportional, are equal.—Therefore also, their like aliquot parts, pyramids or cones ^f, must also be equal ^g.

^a 9. and 12. 6. ^b 2. 7. ^c Hyp. ^d Ax. 7. 4. ^e Cor. 1. 3. 4.

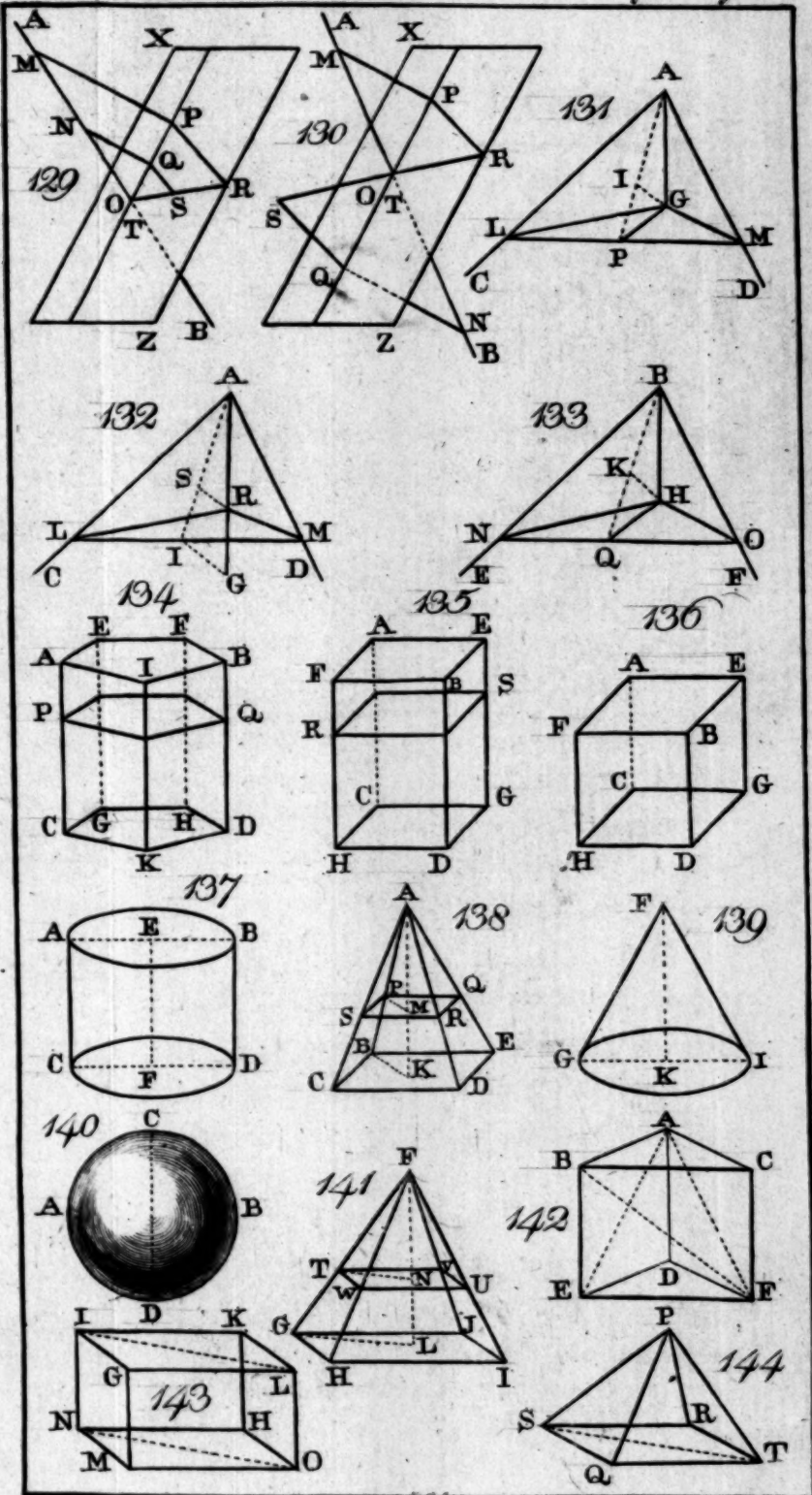
^f 4. 7. ^g Ax. 3. 1.

THEOREM VI.

EVERY sphere is two thirds of a cylinder having the same altitude and diameter as the sphere.

LET ABC be the half of any sphere cut off by a plane passing by a diameter AC (145). Let OB be a radius perpendicular to the section or circle APCQ, which circle conceive to move along OB, always perpendicular thereto, till the point O arrive at B: so will a cylinder ADEC be formed, having the same altitude and diameter as the hemisphere. Let OE be joined; and, from any point, F, in OB, let FG be perpendicular to OB, meeting OE in H, BC in I, and EC in G: also, let OI be joined.—Then, because angle FHO = BEO^a = FOH^b, FO = FH^b; also OI = OC = FG^c. Hence, a circle described with the radius FO (or FH) is equal to the difference of the two circles described with the radii FI and OI (or FG)^d. Now, if we conceive OE to move round the circle DRES, the extremity, O, remaining fixed, it will describe a cone ODE; and if FG be conceived to move round OB, always perpendicular thereto, it will describe circular sections of the cone, hemisphere, and cylinder, with FH, FI, FG, all at the same altitude respectively, and parallel to the

^a Cor. 2. 3. 1. ^b 6. 1. ^c Hyp. ^d Cor. 2. 12. 5.





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base DRES. But (as has been already proved) a circle described with FH, is equal to the difference of the two circles described with FI, FG. Therefore the section of the cone is equal to the difference of the sections of the hemisphere and cylinder, or to the section of the excess of the cylinder above the hemisphere, made by IG: and these two sections being thus equal at every the same altitude, the solids themselves are equal^a; that is, the cone is equal to the excess of the cylinder above the hemisphere. But the cone is one third of the cylinder^b, consequently, the hemisphere ABC is two thirds of the cylinder ADEC. Therefore the whole sphere is two thirds of twice that cylinder, or of a cylinder having the same altitude and diameter as the sphere^c.

COR. HENCE, a cone, hemisphere or sphere, and cylinder, having the same altitude and diameter, are to one another as the numbers 1, 2, 3, respectively.

^a Ax. to 7. ^b Cor. 1. 4. 7. ^c Cor. 1. 2. 4.

THEOREM VII.

PRISMS and cylinders, as also pyramids and cones, having their bases to one another as the squares of their altitudes, are themselves to one another in the triplicate ratio of their altitudes, or as the cubes of their altitudes; that is, as cubes having their dimensions equal to the altitudes of the given solids: Also, all spheres are to one another in the triplicate ratio of their diameters or radii, or as the cubes of their diameters or radii.

LET ABCD, EFGH (146, 147) represent two prisms, having base CD : base GH :: alt. Alsq. : alt. EKsq.; let AI : EK : Q : R; and let LMNO (148) be another prism having its base NO=CD, and altitude LP=R.—Then, CD : GH :: Alsq. : EKsq.^a :: EKsq. : Qsq.^b :: EK : R^c. Therefore NO : GH :: EK : LP^d; and consequently, EFGH=LMNO^e; whence, ABCD : EFGH :: ABCD : LMNO^d :: AI : LP or R^f. —But, the cubes of AI, EK (having their bases, Alsq., EKsq., to one another as the squares of their altitudes AI, EK) are (by what has been just

^a Hyp.^b Cor. 8. 5.^c Cor. 4. 9. 5.^d Ax. 7. 4.^e 5. 7. ^f 2. 7.

proved) to one another in the triplicate ratio of AI, EK. Therefore also, $ABCD : EFGH :: \text{cube of AI} : \text{cube of EK}^a$.—The method of demonstration is the same, whether we consider the figures as representing prisms or cylinders, pyramids or cones : or, since the proportion holds with regard to prisms, it must hold with regard to cylinders, they being equal to prisms of the same bases and altitudes^b ; and also with regard to pyramids and cones, which are like aliquot parts of prisms or cylinders of the same bases and altitudes^c.—And, since spheres are like aliquot parts of cylinders of the same diameters and altitudes as the spheres^d, and those cylinders (having their bases to one another as the squares of their diameters or altitudes^e) are (as has just been shewn) in the triplicate ratio of their diameters or altitudes, or as the cubes of those lines ; the spheres are also in that ratio^f.

COR. 1. THE cubes of four proportional lines (A, B, C, D) are themselves proportional. For, by this Theorem, it appears, that the cube of A is to the cube of B in the triplicate ratio of A, B ; and the cube of C to the cube of D in the triplicate ratio of C, D. But (because $A : B :: C : D$) these ratios are equal^g. Therefore, cube of A : cube of B :: cube of C : cube of D^h.

COR. 2. Prisms or pyramids, having similar bases, and having also the corresponding sides of

^a Ax. 6. 4. ^b 1. 7. ^c Cor. 1. 4. 7. ^d 6. 7. ^e 12. 5.

^f Cor. 1. 2. 4. ^g 5. 4. ^h Ax. 6. 4.

the bases to one another as the altitudes, are themselves to one another in the triplicate ratio of their altitudes or of the corresponding sides of the bases, or as the cubes of those lines. Thus (146, 147) if CD, GH be similar, CS, GT , corresponding sides, and $CS : GT :: AI : EK$, then $CD : GH :: CS^2 : GT^2 :: AI^3 : EK^3$. Therefore the solids themselves (whether we consider them as prisms or pyramids) are to one another in the triplicate ratio of AI, EK , or as the cubes of AI, EK , and, consequently, also in the triplicate ratio of CS, GT , or as the cubes of CS, GT .

COR. 3. Prisms or pyramids bounded wholly by planes respectively similar, are to one another in the triplicate ratio of their altitudes, or of the corresponding sides of the similar planes, or as the cubes of those lines. Thus (146, 147) supposing $AC, EG; CS, GT$, corresponding sides, $CD : GH :: CS^2 : GT^2 :: AC^3 : EG^3 :: AI^3 : EK^3$. Therefore the solids themselves (whether we consider them as prisms or pyramids) are to one another in the triplicate ratio of AI, EK , or as the cubes of AI, EK ; and, consequently, also in the triplicate ratio of AC, EG , or of CS, GT , or as the cubes of those lines.

COR. 4. Cylinders or cones (149, 150) having the diameters (AB, CD) of the bases to one another as the altitudes (EF, GH) are themselves

^a 9. 5. ^b Cor. 8. 5. ^c 7. 7. ^d 5. 4. ^e Cor. 1. 7. 7.

^f 15. 6. and Cor. 8. 5.

to one another in the triplicate ratio of their altitudes or of the diameters of the bases, or as the cubes of those lines: because, base $IK : LM : ABsq. : CDsq.^a :: EFsq. : GHsq.^b$.

COR. 5. Cylinders or cones (149, 150) having the diameters (AB, CD) of the bases to one another as the axes (EN, GO) and the angles (NEF, OGH) of the axes with perpendiculars (EF, GH) let fall from their extremities upon the planes of the bases equal, are to another in the triplicate ratio of their altitudes, axes, or diameters of the bases, or as the cubes of those lines: for the triangles NEF, OGH being equiangular^c; we have base $IK : base LM :: ABsq. : CDsq.^a :: ENsq. : GOsq.^b :: EFsq. : GHsq.^d$.

^a 12. 5. ^b Cor. 8. 5. ^c Cor. 5. 4. 1. ^d 4. 5. and Cor. 8. 5.

SECTION VIII.

PROBLEMS RESPECTING LINES.

PROBLEM I.

FROM any point (A) as a center, to describe the circumference of a circle, with a radius equal to any given line (BC).
—(151).

TAKE the given line in a pair of compasses, that is, extend their points from one extremity, B, to the other C; then, fixing one point of the compasses in A, make the other point, with that extent, revolve quite round, describing thereby the circumference DEF. DEF is the circumference required, by construction.

COR. HENCE the manner of describing a circle, semi-circle, or arch, with any radius, is obvious.

P R O B L E M II.

TO produce a given line (AB) from either extremity (B) at pleasure.—(152).

APPLY the edge of a straight ruler to AB, so as that a part of the ruler extend beyond B; then, with the point of a pen or pencil, along the edge of the ruler, draw a line beyond B to any point C.—AB is produced to C, by construction.

P R O B L E M III.

IN the greater of any two given lines (AB, CD) to take a part equal to the less (AB).—(153).

TAKE AB in your compasses; then, fixing one point in either extremity, C, of the other line, describe an arch, cutting CD in E. CE is equal to AB, by construction.

COR. HENCE, the method of drawing a line from any point, equal to any given line, is manifest.

P R O B L E M IV.

TO draw a line from one given point (A) to another (B).—(153).

APPLY the edge of a straight ruler to both points ; then, with the point of a pen, along the edge of the ruler, draw a line, AB. AB is the line required, by construction.

COR. HENCE the method of drawing a line through one or two given points, is manifest *.

* N. B. It is not thought necessary to refer to the foregoing Problems in what follows.

P R O B L E M V.

FROM any point (A) in a given line (BC) to draw a perpendicular to that line.—(154).

IN BC (produced if necessary) take any two equal distances, AD, AE ; and, from the points D, E, with any radius greater than AD or AE, describe the circumferences of two circles, which will cut one another, because, by construction,

part of either circle falls within, and part without the other ; and, to the point of intersection, F, draw AF. AF is perpendicular to BC.—For, because $DF=EF^a$, angle $ADF=AEF^*$; whence, AD being also $=AE^a$, angle $DAF=EAF^b$.—Tis obvious, that only so much of the circumferences need be described as will give the point of intersection.

Another Method.

FROM any point, D, out of BC (155) describe the circumference or semi-circumference of a circle, cutting BC in A, and in any other point, E; draw the diameter EDF, and to the point F, draw AF. BAF is a right angle^c.

COR. FROM the former method, it appears, that circles described from two points, with a radius greater than half the distance of those points, will cut each other.

* This will readily appear without referring to Theor. 6 I, by conceiving the part of triangle FDE which is towards D turned over upon that part which is towards E, so as that FD coincide with its equal FE; for then, the part of DE which is towards D, must fall upon the other part towards E^d, because their extreme points coincide.

^a Conf. ^b I. I: ^c Cor. 5. 3. 3. ^d Ax. 7. I.

PROBLEM VI.

FROM any point (F) out of a given line (BC) to draw a perpendicular to that line —(156).

FROM F, describe an arch of a circle, so as to cut BC (produced if necessary) in two points, D, E; from which points, with any radius greater than half their distance DE, describe two other arches, cutting each other in G^a; and draw FG, cutting BC in A. FA is perpendicular to BC.—For, drawing FD, FE, GD, GE, the triangles DFE, DGE, are isosceles^b; and, consequently, angle FDE=FED and GDE=GED^c. Therefore angle FDG=FEG^d. Whence, angle AFD=AFE^e; and, consequently, angle FAD=FAE^e.

Another Method.

FROM any point, D, in BC (157) describe an arch through F, intersecting BC in E; with the distance EF, from E describe another arch, cutting the former in G; and draw FG, cutting BC in A. FA is the perpendicular required.—For, drawing DF, DG, EF, EG, the demonstration is similar to the foregoing.

^a Cor. 5. 8. ^b Conf. ^c Note to Prob. 5. ^d Ax. 3. 1.
^e 1. 1.

P R O B L E M VII.

TO bisect any given angle (ABC).—
(158).

FROM the angular point, B, with any radius, describe an arch, cutting BA, BC in two points, D, E; from which points, with any radius greater than half their distance DE, describe two other arches, cutting each other in F; and draw BF. BF bisects ABC.—For, drawing DF and EF, triangles BDE, FDE are isosceles^a; and, consequently, angle BDE=BED and FDE=FED^b. Hence, angle BDF=BEF^c; and, consequently, angle DBF=EBF^d.

P R O B L E M VIII.

TO bisect any given line (AB).—
(159).

FROM the extreme points, A, B, with any radius greater than half the given line, describe two arches intersecting each other in C, D; and draw CD, cutting AB in E. AE is equal to BE.—For, drawing AC, AD, BC, BD, triangles ACB,

^a Conf. ^b Note to Prob. 5. ^c Ax. 3. 1. ^d 1. 1.

ADB, are isosceles ^a; consequently, angle CAB = CBA, and DAB = DBA ^b. Hence, angle CAD = CBD ^c; and, therefore, angle ACE = BCE ^d; whence AE = BE ^d.

COR. FROM the last step it appears, that CD not only bisects AB, but is perpendicular to it.

PROBLEM IX.

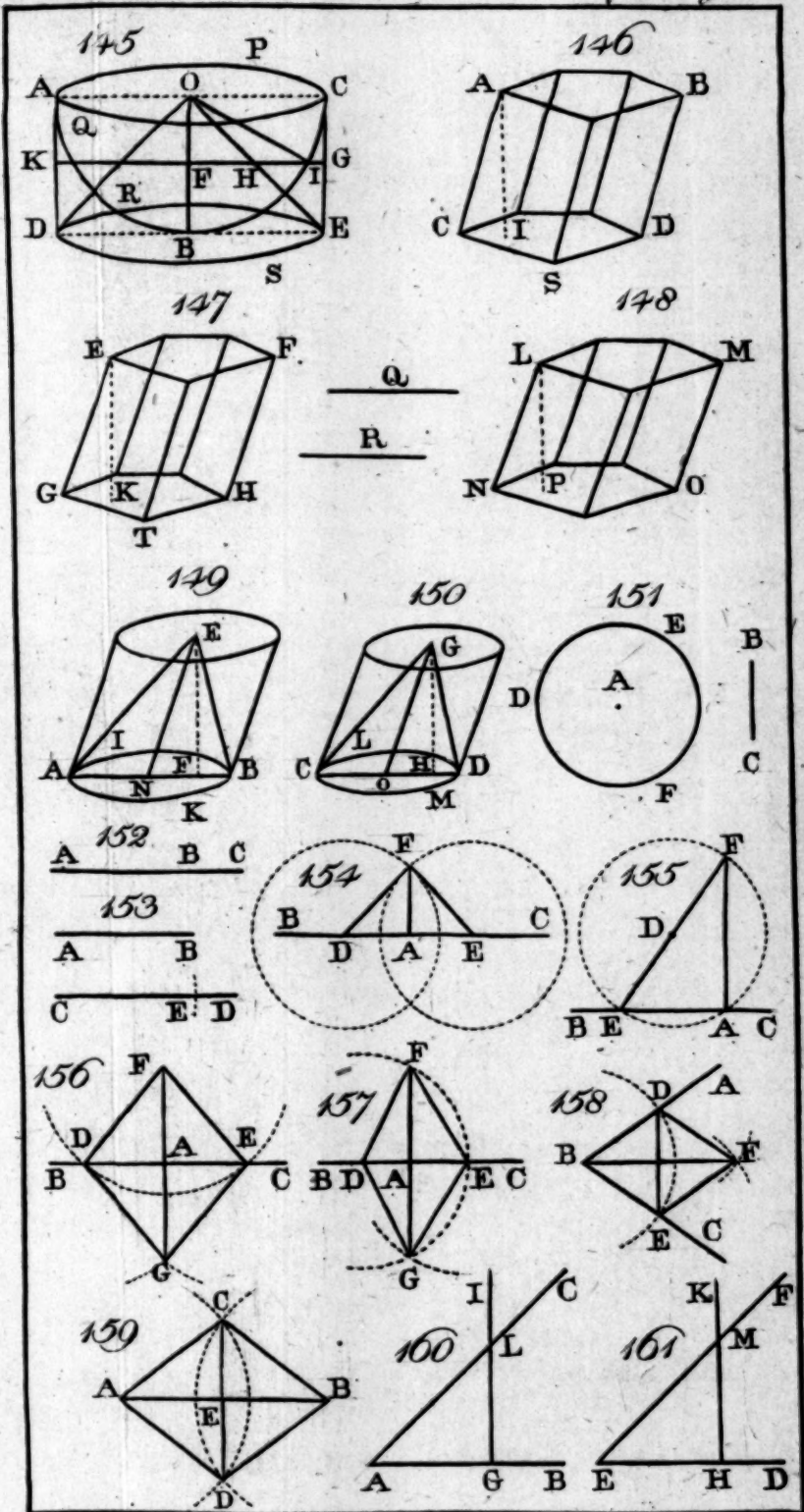
FROM any point (A) in a given line (AB) to draw another line making therewith an angle equal to any given angle (E).—(160, 161).

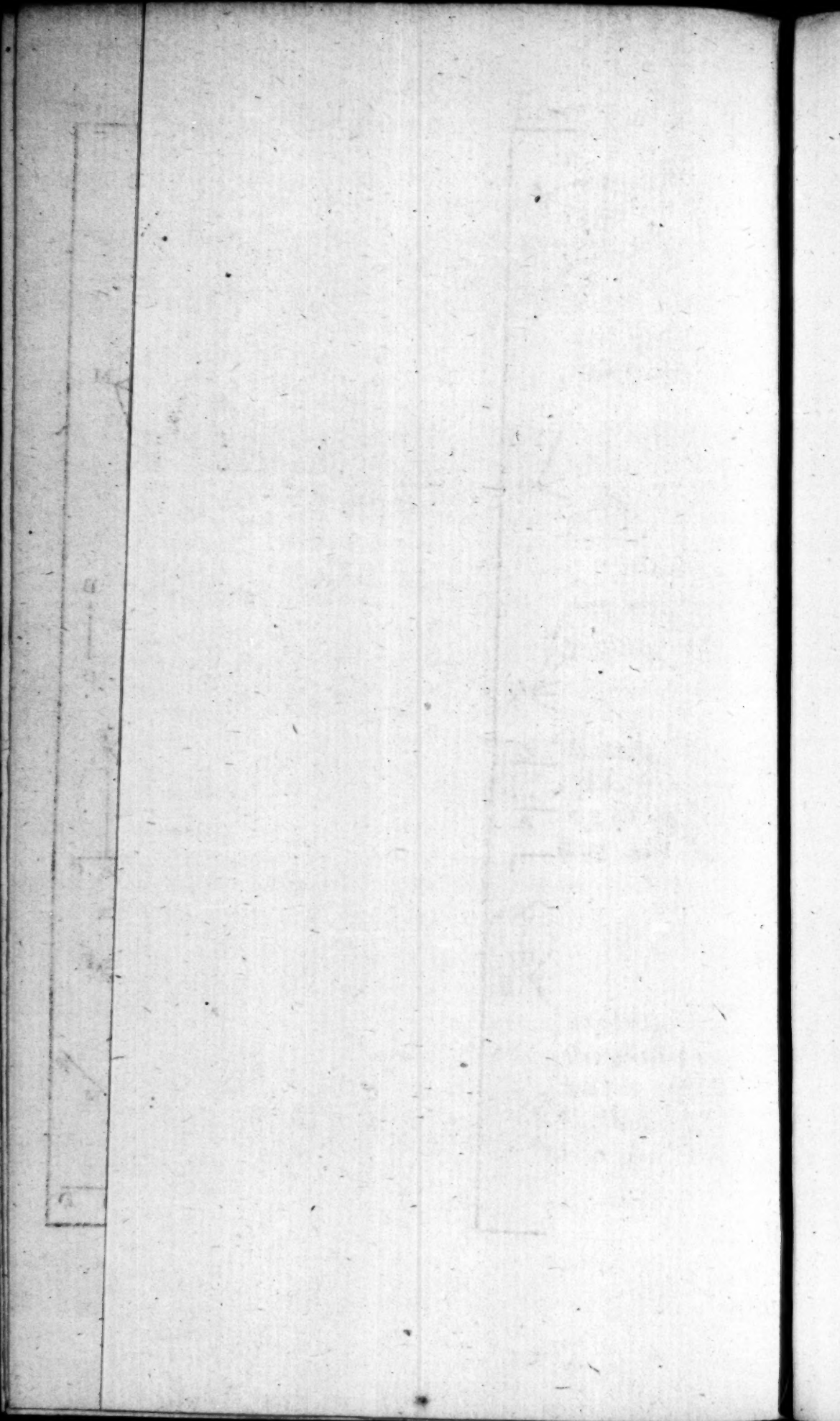
IN AB, ED, take two equal distances, AG, EH; at the points G, H, draw GI, HK, perpendicular to AB, ED, the latter cutting EF in M; take GL = HM; and, through L, draw AC. The angle A = E ^d.

Another Method.

FROM the points E and A (162, 163) with any radius, describe two arches, cutting the given lines in G, H, I; from the point I, with a radius equal to the distance of the points G, H, describe another arch cutting the former in K; and

^a Conf. ^b Note to Prob. 5. ^c Ax. 3. 1. ^d 1. 1.





through K draw AC. The angle $A=E$.—For, drawing the straight lines GH, IK, the three sides of triangle AIK are respectively equal to the three sides of triangle EGH^a. Therefore, angle $A=E$ ^b.

P R O B L E M X.

THROUGH any given point (A) to draw a line parallel to a given line (DE).—(164).

FROM A, to any point, F, in DE, draw AF; through A draw BC, making angle $BAF=EFA$ ^c. BC is parallel to DE^d *.

Another Method.

TAKE any point, F, in DE; make FG equal to the distance of the points AF; from A, G, with the same distance, describe two arches cutting each other in a point H (they will cut each other, because the sum of their radii, AF, FG, is greater, and their difference less than the distance of their centers AG^e); and, through A, H, draw BC. BC is parallel to DE, because, in the quadrangle AFGH the sides are all equal^f.

* Thus, in Problems 2. 3. 4. 5. 6. 7. 9. and 10. the operations mentioned in the Postulates, Sect. 1. are effected without referring to those Theorems in which they were required §.

^a Conf. ^b 11. 1. ^c 9. 8. ^d 3. 1. ^e Cor. 4. 7. 3. and 9. 1.
^f 3. 2. and Conf. § See note page 4th.

PROBLEM XI.

TO divide any line (AB) into parts, which shall have the same ratio to the whole (AB), as the corresponding parts of another line (QR) any how divided, have to the whole (QR).—(165).

FROM the points A, B, on different sides of the given line, draw two lines, AC, BD, parallel to each other^a; in AC, take AE, EF, FG, GH, equal respectively to QS, ST, TV, VR; in BD, take BI, IK, KL, LM, equal respectively to RV, VT, TS, SQ; and draw EL, FK, GI, intersecting AB in N, O, P. AB is divided as was required.—For, because EF is parallel and equal to LK, EN is parallel to FO^b; and, in the same manner is FO parallel to GP, and GP to HB. Hence $AN : AB :: AE$ or $QS : AH$ or QR , &c.^c.

COR. 1. IF the parts AE, EF, &c. BI, IK, &c. were all equal, it appears that the parts AN, NO, &c. would also be all equal; whence, a method of dividing any line into any number of equal parts is manifest.

COR. 2. Hence also (and from Theor. 1. 5.) a method of dividing any triangle or parallelogram into any number of parts, each of which

^a 10. 8. ^b 3. 2. ^c Cor. 3. 5.

shall have a given ratio to the whole, is manifest; namely, by dividing the base as above, and from the several divisions drawing lines to the vertex of the triangle, or to the opposite side of the parallelogram; those lines, in the latter case, being drawn parallel to the other sides.

PROBLEM XII.

TO find a third-proportional to two given lines (A, B).—(166).

FROM any point, C, draw two lines, CP, CQ, making any angle; take $CD=A$, $CE=B$, and CF likewise $=B$; join DE ; and draw FG parallel to DE^a . CG is the proportional required.—For CD or $A : CE$ or $B :: CF$ or $B : CG^b$.

PROBLEM XIII.

TO find a fourth-proportional to three given lines (A, B, C).—(167).

FROM any point, D, draw two lines, DP, DQ, making any angle; take $DE=A$, $DF=B$, and $DG=C$; join EF ; and draw GH parallel to EF^a . DH is the proportional required.—For, DE or $A : DF$ or $B :: DG$ or $C : DH^b$.

^a 10. 8. ^b 3. 5.

PROBLEM XIV.

TO find a mean-proportional between two given lines (A, B).—(168).

DRAW a line, PQ; from any point in which, take $CD=A$, and $CE=B$; bisect DE in F ; from which point, with the radius FD or FE , describe the semi-circle DGE ; and draw CH perpendicular to DE , and meeting the arch in H . CH is the proportional required^a.

PROBLEM XV.

TO divide a given line (AB) in extreme and mean proportion: that is, to divide it into two parts, such that the whole line shall be to the greater part, as the greater part to the less; or, such that the square of the greater part shall be equal to a rectangle under the whole line and the less part.—(169).

DRAW BC perpendicular to AB and equal to $\frac{1}{2}AB$; join AC , in which take $CD=CB$ or $\frac{1}{2}AB$,

^a Cor. 7. 5.

and in AB take $AE=AD$. I say $AB : AE : BE$,
or $AEsq.=rect. AB, BE$.—For, by substituting
of equals, we have, $AEsq.=ADsq.^a=ACsq.-$
 $CDsq.-2 rect. AD, CD^b=ABsq.+BCsq.^c-$
 $BCsq.^a-2 rect. AE, \frac{1}{2}AB^d=ABsq.-rect. AB,$
 $AE,=rect. AB, BE+rect. AB, AE^e.-rect. AB,$
 $AE,=rect. AB, BE$. Whence also, $AB : AE :$
 BE^f .

^a Cor. 5. 1. 2. ^b Cor. 4. 6. 2. ^c 7. 2. ^d Cor. 4. 1. 2.
^e Cor. 2. 5. 2. ^f Cor. 3. 2. 5.

L 3 SEC.

SECTION IX.

PROBLEMS RESPECTING SURFACES.

PROBLEM I.

THREE lines (A, B, C) being given, of which the sum of any two is greater than the third, 'tis required to construct a triangle having its sides respectively equal to the given lines.—(170).

MAKE $DE = A$; from the points D, E, with radii equal to B, C, describe two arches intersecting one another in a point F (they will intersect each other, for $A > B + C$ and (because $A + C < B$) $A < B - C$); and draw DF, EF. DEF is the triangle required, by construction.

COR. 1. HENCE the difference of any two sides of a triangle is less than the third.

COR. 2. In the same manner may a triangle be made equal in every respect to any given triangle.

P R O B L E M II.

UPON a given line (AB) to construct an equilateral or equiangular triangle.—(171).

FROM the points A and B, with the radius AB, describe two arches intersecting one another at C; and draw AC, BC. The triangle ABC is both equilateral and equiangular ^a.

P R O B L E M III.

UPON a given line (AB) to construct an isosceles triangle, having each of its equal sides equal to a given line (D) greater than half the base (AB).—(172).

FROM the points A and B, with the radius D, describe two arches cutting each at C; and draw AC, BC. ABC is the triangle required ^b.

^a Conf. and Cor. 1. 6. 1. ^b Conf.

P R O B L E M IV.

TO construct a right-angled triangle, having the hypotenuse (A) and another side (B) given; or, having the sides (C, D) to contain the right angle given.—(173, 174).

PART I. MAKE $EF=A$ (173); bisect EF in G^a ; with the radius GE or GF describe a semi-circle EHF ; with the radius B (which must be less than A or EF^b) describe an arch cutting the arch of the semi-circle in I ; and join EI , FI . EFI is the triangle required c .

PART II. MAKE $LM=C'$ (174); on either extremity raise a perpendicular, MN , equal to D^d ; and join LN . LMN is the triangle required e .

P R O B L E M V.

TO construct a parallelogram, having one of its angles (ABC) and the sides (AB , CB) containing that angle given.—(175).

a 8. 8. b 7. 2. c Conf. and Cor. 5. 3. 3. d 5. 8. e Conf.

FROM A with the radius CB, and from C with the radius AB, describe two arches cutting each other in a point D, (they will cut each other, because the sum of their radii $(CB+AB)$ is greater, and their difference $(CB-AB)$ less than AC^a); and draw AD, CD.—ABCD is the figure required^b.

COR. HENCE the manner of constructing a square upon any given line, or a rectangle under any two given lines, is manifest: for, if we suppose B a right angle, ABCD would be a rectangle; and if we suppose $AB=CB$, the same figure would be a square.

P R O B L E M VI.

UPON a given line (AB) to make a rectangle equal to any given straight-lined figure (CDEF).—(176, 177).

DIVIDE the given figure into triangles, CFD, DFE; to any side in each triangle, from the opposite angle, draw a perpendicular^c; at either extreme of AB, raise a perpendicular, AL^d; in which, take AI, equal to a fourth-proportional to 2 AB, CD, FG; and IL, equal to a fourth-proportional to 2 AB, FD, EH^e; and complete the rectangle LABM^f. $LABM=CDEF$.—Draw IK perpen-

^a 9. 1. and Cor. 1. 1. 9. ^b 3. 2. ^c 6. 8. ^d 5. 8. ^e 13. 8.
^f 5. 9.

dicular to AL. Then (since $2 AB : CD :: FG : AI$) $2 \text{ rect. } AB, AI = \text{rect. } CD, FG^a$; and, consequently, $\text{rect. } AB, AI$ (or AK) $= \frac{1}{2} \text{ rect. } CD, FG^b = \text{triangle } CFD^c$. In the same manner, $\text{rect. } IM = \text{triangle } DFE$. Consequently $LABM = CDEF^b$.

COR. If a parallelogram be required to be made upon AB equal to CDEF, and having an angle equal to a given angle, P, make angle $BAN = P$, and draw BO parallel to AN meeting LM in O, so will NB be the parallelogram required^d.

PROBLEM VII.

TO make a square equal to any given straight-lined figure (CDEF).—(176, 177).

UPON any line, AB, make a rectangle, AM, equal to $CDEF^e$; or, having found AL as in the last, find a mean-proportional, Q, between AB and AL^f. Then $Qsq. = CDEF$.—For (since $AB : Q : AL$) $Qsq. = \text{rect. } AB, AL^g = CDEF^h$.

^a Cor. 2. 2. 5. ^b Ax. 3. 1. ^c Cor. 3. 2. 2. ^d Cor. 2. 2. 2.

^e 6. 9. ^f 14. 8. ^g Cor. 3. 2. 5. ^h Conf.

P R O B L E M VIII.

TO make a rectangle or square equal to the sum or difference of any two given straight-lined figures (A, B). — (178, 179, 180).

ON any line, CD, describe two rectangles, CE, CF, respectively equal to A, B; one of them, CF, on both sides of CD. Rectangle HE is the sum and IE the difference required^a; from which a square equal to the sum or difference may be constructed by the last.

COR. 1. HENCE (as was observed in Cor. to Prob. 6.) a parallelogram may be made upon any given line equal to the sum or difference of any two given straight-lined figures, and having an angle equal to any given angle.

COR. 2. Hence also, two lines may be found having the same ratio to one another as any two straight-lined figures: for $A : B :: CE : CF^b$
 $:: CG : CI^c$.

^a Ax. 3. 1. ^b Ax. 7. 4. ^c I. 5.

PROBLEM IX.

TO make a square equal to the sum or difference of any two squares. — (181).

LET A, B , be the sides of two squares, and make a right angle, P .

PART I. TAKE $PQ=A$, $PR=B$, and join QR .
 $QRsq.=PQsq.+PRsq.=Asq.+Bsq.^a$.

PART II. Make $PQ=A$, the less line; with the radius B , describe an arch cutting PR in S , and join QS . $PSsq.=QSsq.-PQsq.=Bsq.-Asq.^b$.

COR. HENCE, a square may be made equal to the sum of three or more squares. For, if QR , and C , the side of another square, be set off from P , we shall, as above, have an hypotenuse, the square of which will be equal to $QRsq.+Csq.$ or to $Asq.+Bsq.+Csq.^a$.

^a 7. 2. ^b Cor. 1. 7. 2.

P R O B L E M X.

TO describe a figure equal and similar to any given straight-lined figure (ABCDE).—(182, 183).

DIVIDE the given figure into triangles. Make $FG=AB$; on which, make triangle FIG , having $FI=AD$ and $GI=BD$; on FI , make triangle FKI , having $FK=AE$ and $IK=DE$; and, on GI , make triangle GHI , having $GH=BC$ and $IH=DC^a$. $FGHIK$ is equal and similar to $ABCDE^b$.

P R O B L E M XI.

UPON a given line (PQ) to construct a figure similar to any given straight-lined figure (ABCDE).—(182, 184).

DIVIDE the given figure into triangles. Make angles PQS , QPS , respectively equal to ABD , BAD ; angles PST , SPT , respectively equal to ADE , DAE ; and angles QSR , SQR , respectively equal to BDC , DBC^c ; the legs of the angles being produced to meet in the points S , T , R , will form a figure $PQRST$, similar to $ABCDE$.—For, angle $PQR=ABC^d$, $R=C^e$, &c. also $AB:PQ$ ($:: DB:SQ$) $:: BC:QR :: DC:SR$ &c. f .

^a I. 9. ^b II. 1. ^c 9. 8. ^d Ax. 3. 1. ^e Cor. 5. 4. 1.

^f 4. 5.

P R O B L E M XII.

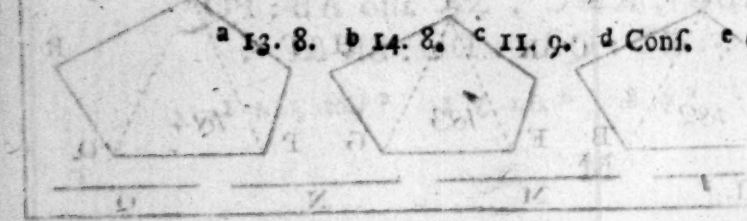
ANY straight-lined figure (ABCDE) being given, 'tis required to construct a figure similar to it, and to which the figure given shall have the same ratio as any given line (L) to any other given line (M).—(182, 184).

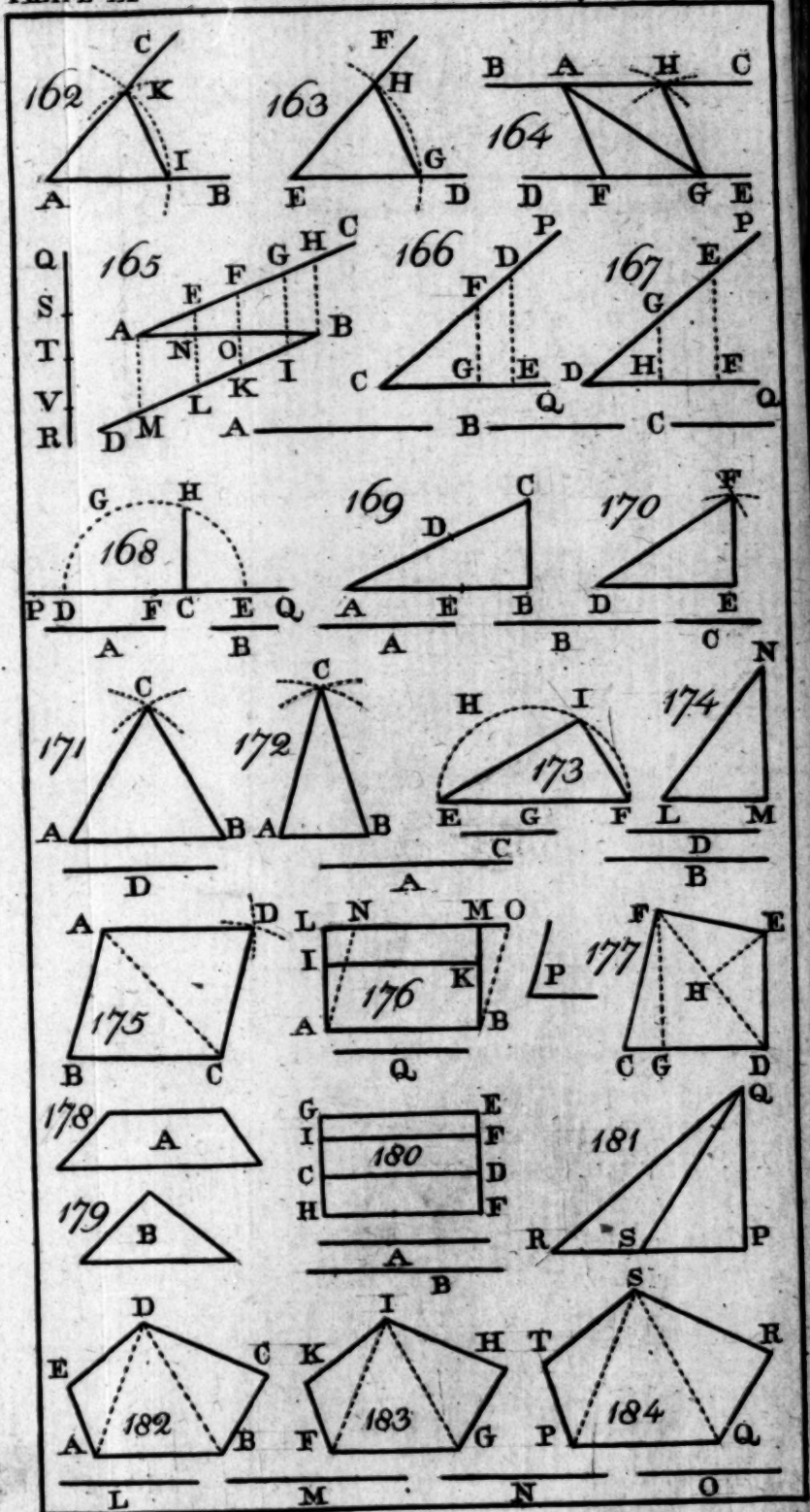
To L, M, AB, find a fourth-proportional, N^a ; and between AB and N, find a mean-proportional, O^b ; make $PQ=O$, and construct thereon PQRST similar to ABCDE^c. PQRST is the figure required.—For (since $AB : PQ$ or $O : N^d$) $ABCDE : PQRST :: AB : N^e :: L : M^d$.

P R O B L E M XIII.

TO make a figure equal to one straight-lined figure (A) and similar to another straight-lined figure (B).—(185, 186, 187, 188).

^a 13. 8. ^b 14. 8. ^c 11. 9. ^d Conf. ^e Cor. 4. 9. 5.







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DRAW $EF=CD$; on which, make rectangle $EH=B$; and, on FH , make rectangle $FI=A^a$; between EF, FK , find a mean-proportional, LM^b ; on which, make the figure N , similar to B^c . I say N is also equal to A .—For (since EF or CD : LM : FK^d) $B:N::EF:FK^e::EH:FI^f::B:A^g$. Therefore $N=A^h$.

P R O B L E M XIV.

TWO straight-lined figures (A, B) being given; 'tis required to construct a third, to which the first figure (A) shall have the same ratio as any line (P) to any other line (Q), and which third figure shall also be similar to the second (B).—(185, 186, 187, 189).

DRAW $EF=CD$; on which, make rectangle $EH=B$; and, on FH , make rectangle $FI=A^a$; to P, Q, FK , find a fourth-proportional, S^i ; make $FT=S$, and complete rectangle FV^k ; then, between EF, FT , find a mean-proportional, WX ; on which, make the figure Y similar to B^c . I say also, that $A:Y::P:Q$.—For, by Theor. 13. it appears that $Y=FV$. Consequently, $A:Y::FI:FV^g::FK:FT$ or $S^f::P:Q^l$.

^a 6. 9. ^b 14. 8. ^c 11. 9. ^d Conf. ^e Cor 4. 9. 5. ^f 1. 5.
^g Ax. 7. 4. ^h Cor. 1. 3. 4. ⁱ 13. 8. ^k 5. 9. ^l Conf.

PROBLEM XV.

TO find the center of a circle (AB).—
(190).

DRAW any chord CD, and bisect it by a perpendicular FF^a, meeting the circumference both ways; bisect also EF in G. G is the center ^b.

PROBLEM XVI.

FROM a given point (A) to draw a tangent to a circle (DEF).—(191).

I. If the point A be in the circumference; draw the radius OA, perpendicular to which draw AB. AB is a tangent ^c.

II. If the point A be without the circumference; draw OA, which bisect in C ^b; from C, with the radius CA or CO, describe a semi-circle cutting the circumference in D; and join AD. AD is a tangent ^c; because, drawing the radius OD, ODA is a right angle ^d.

^a 8. 8. ^b 4. 3. ^c 6. 3. ^d Cor. 5. 3. 3.

P R O B L E M XVII.

UPON a given line (AB) to describe the segment of a circle which segment shall contain an angle equal to a given angle (P).—(192).

MAKE angle $ABC = P^a$; draw BE perpendicular to CB^b; make angle $BAF = ABE^a$; and, from the point of meeting, F, with the radius AF or BF, describe the segment AGB. AGB is the segment required. For angle $AGB = ABC^c = P^d$.

P R O B L E M XVIII.

TO describe a circle about a given triangle (ABC)*.—(193).

BISECT any two sides, AB, BC, by two perpendiculars^e. These two perpendiculars, DE, FE, must meet in a point, E; because, drawing DF, angle $EDF + EFD$ is less than two right angles^f. The point E is also equidistant from the points

* A circle is said to be *described about* a straight-lined figure, or a straight-lined figure *inscribed within* a circle, when the circumference passes through all the angular points of the figure.

^a 9. 8. ^b 5. 8. ^c 6 and 8. 3. ^d Conf. ^e 8. 8.

^f Ax. 1, 1. and Cor. 2. 3. 1.

A, B, C, as is manifest by drawing AE, BE, CE^a. Therefore a circle described from E, with the radius AE, will pass through the angular points A, B, C^b; which was required.

COR. 1. HENCE it appears, that through any three points, A, B, C, not situated in the same straight line, the circumference of a circle may be described; and the method of performing it is manifest.

COR. 2. Hence, if an arch of a circle or any three points in the circumference be given, the center may be found and the circle completed.

PROBLEM XIX.

TO inscribe a circle in a given triangle (ABC) *.—(194).

BISECT any two of the angles, A, C, by the lines AD, CD, meeting one another in D^c; draw DE perpendicular to AC^d. A circle, described with the radius DE, will touch all the sides of the triangle.—For, drawing DF, DG, perpendicular to AB, BC, it appears that DF, DG, are each

* A circle is said to be *inscribed within* a straight-lined figure or a straight-lined figure *described about* a circle when the circle is touched by all the sides of the figure.

^a Conf. and 1. 1. ^b Def. 1. 3. ^c 7. 3. ^d 6. 3.

equal to DE^a ; therefore the circumference passes through the points E, F, G^b ; and it touches the sides in those points, because the radii are perpendicular to the sides at those points c .

P R O B L E M XX.

IN a given circle (ABC) to inscribe a triangle, having its angles respectively equal to the angles of a given triangle (DEF).—(195, 196).

DRAW the radii OG, OC, OH , making angles COG, COH , each equal to D^d ; draw GH ; make angle $GHI = F^d$; and draw GI . GHI is the triangle required.—For, angle $GHI = F^e$, angle $GIH = GOC^f = D^e$, and angle $IGH = E^g$.

P R O B L E M XXI.

ABOUT a given circle (ABC) to describe a triangle, having its angles respectively equal to the angles of a given triangle (DEF) —(197, 198).

PRODUCE EF both ways; draw the radii OA, OC, OB , making angle $AOC = DEP$ and $BOC =$

^a Cor. 5. 1. ^b Def. 1. 3. ^c 6. 3. ^d 9. 8. ^e Conf.

^f Cor. 1. 3. 3. ^g Cor. 5. 4. 1.

DFQ^a; and draw GI, GH, IH, touching the circle in the points A, C, B^b: then, joining AC, it appears that GI, GH are not parallel, because angle GAC+GCA is less than two right angles^c; and, for a similar reason GH, IH, as also GI, IH, are not parallel; consequently, these lines will meet and form a triangle GHI, which is the triangle required.—For all its sides touch the circle^d; and, because the angles at A, C, B, are right angles, angle G+AOC=2 right angles^e=DEF+DEP^f; whence (subtracting the equals AOC, DEP^d) angle G=DEF^g: in the same manner, H=DFE; whence also, I=D^h.

PROBLEM XXII.

IN a given circle (ABCD) to inscribe a square.—(199).

DRAW two diameters, EF, GH, perpendicular to each otherⁱ; and draw EG, GF, FH, HE. EGFH is the square required.—For the sides are all equal^k. and the angles right angles^l.

COR. IF diameters be drawn bisecting the angles at the center, and their extreme points be joined, a regular octagon will be inscribed in the circle; and, by bisecting the angles at the center

^a 9. 8. ^b 16. 9. ^c Ax. 1. and Cor. 2. 3. 1. ^d Conf.

^e Cor. 6. 4. 1. ^f 2. 1. ^g Ax. 3. 1. ^h Cor. 5. 4. 1.

ⁱ 5. 8. ^k 1. 1. ^l Cor. 5. 3. 3.

again, a regular polygon of double that number of sides will be inscribed, &c.—Thus may a circle and its circumference be divided into 2, 4, 8, 16, 32, &c. equal parts *.

P R O B L E M XXIII.

IN a given circle (ABCD) to inscribe a regular pentagon or decagon.—(200).

DRAW a diameter, AC; from the center, E, draw a radius, EB, perpendicular to AC; bisect AE in F, and make FG=FB. Then BG=the side of the pentagon, and EG=the side of the decagon; from which the figures themselves may be described.—Suppose AH a side of the required pentagon, and HI a side of the decagon; by drawing AI, EI, EH, and AK perpendicular to EH, it will appear that BG=AH and EG=HI. For, in triangle LEI (because angle HEI = $\frac{1}{2}$ CEI^a = CAI^b = AIE^c) LI=LE^c; whence, in triangle ALE, angle ALE (=AIE+HEI^d) = 2 HEI = AEH^a, and, consequently also, AL=AE^c; whence again, in triangle HIL (because

* Straight-lined figures of more than four sides, are called by the general name of *Polygons*, and are said to be *regular* or *irregular*, according as their sides and angles are all equal or not. They have also particular names according to the number of their sides. A *Pentagon* has 5 sides; a *Hexagon*, 6; a *Heptagon*, 7; an *Octagon*, 8; a *Nonagon*, 9; a *Decagon*, 10; &c.

^a 10. 5. ^b 3. 3. ^c 6. 1. ^d Cor. 1. 4. 1.

angle $HLI = ALE^a$, and $AIH = CAI^b$, and, consequently, $IHL = AEH^c = ALE = HLI$) $HI = LI^d = LE$. But, triangles ALE , HIL , being proved mutually equiangular, we have $AE : HI :: LE : HL^e$, and, consequently, $rect. AE, HL = rect. HI, LE^f$, or (because $EH = AE$ and $LE = HI$) $rect. EH, HL = LEsq.^g$. But (because $FE = \frac{1}{2} EB = \frac{1}{2} EC$ and $FG = FB$) it appears, that $rect. EC, CG = EGsq.^h$, whence (EH being $= EC$) 'tis manifest $EG = LE = HI$.—And, since AL has been proved equal to AE , $KE = KL^i$; consequently $KH - KE = KH - KL^k = HL$. Hence $AHsq. - AEsq. = rect. HE, HL^l = LEsq.$ Consequently, $AHsq. = LEsq. + AEsq.^k = EGsq. + BEsq.^m = BGsq.^n$. Therefore $BG = AH^o$.

COR. HENCE a circle and its circumference may be divided into 5, 10, 20, &c. equal parts.

PROBLEM XXIV.

IN a given circle (ACF) to inscribe a regular hexagon.—(201).

THE side of the figure required, is equal to the radius of the circle; whence the figure itself may

^a Cor. 5. 2. 1. ^b Cor. 2. 3. 3. ^c Cor. 5. 4. 1. ^d 6. 1.

^e 4. 5. ^f Cor. 2. 2. 5. ^g Cor. 4. 1. 2. ^h 15. 8. ⁱ Cor. 12. 1.

^k Ax. 3. 1. ^l Cor. 2. 7. 2. ^m Cor. 5. 1. 2. ⁿ 7. 2.

^o Cor. 5. 6. 2.

be described.—For, let AB be a side of the required hexagon. Draw the radii AG, BG. Then (since angle $AGB = \frac{1}{2}$ of 4 right angles^a $= \frac{1}{3}$ of 2 right angles) angle $GAB + GBA = \frac{2}{3}$ of two right angles^b. But angle $GAB = GBA$ ^c. Therefore angle $GAB = \frac{1}{3}$ of two right angles = angle AGB . Consequently $AB = BG$ ^c.

COR. IF BC be taken equal AB, AC will be the side of an equilateral triangle inscribed in the circle.—Hence, a circle and its circumference may be divided into 3, 6, 12, &c. equal parts.

P R O B L E M XXV.

TO inscribe a circle within, or to describe a circle about, any regular polygon (ABCDEF).—(202).

BISECT any two adjoining sides, AF, FE, with two perpendiculars, GO, HO^d. These, not being parallel^e, must meet in a point, O. That point is the center of the circle required; the radius of the inscribed circle is OG, and that of the circumscribed circle is OF.—For, to the middle of the sides, draw OI, OK^d, &c. and join GH, HI &c. Then, in triangles FGH, EHI, &c. angle $FGH = FHG = EHI = EIH$ &c. also $GH = HI$

^a 10. 5. ^b 4. 1. ^c 6. 1. ^d 8. 8. ^e See Dem. of 21. 9.

&c.^a. From angles OGF, OHF, take FGH, FHG; there remains, angle OGH=OHG^b: consequently OH=OG^c. From angles OHF, OHE, take FHG, EHI; there remains, angle OHG=OHI^b; whence OI=OG^d; also, angle OIH=OGH^d, which, added to angles EIH, FGH, gives angle OIE=OGF^b=a right angle. In this manner it appears that OG, OH, OI, OK, &c. are equal to one another and perpendicular to the sides of the given figure. Therefore a circle described from O with the radius OG will touch the sides of the given figure in the points G, H, I, &c.^e. —And, by drawing OF, OE, OD, &c. it appears that OF=OE=OD &c.^d. Therefore, also, if a circle be described from O with the radius OF, the circumference will pass through the points F, E, D, &c.

COR. 1. SINCE OF, OE, bisect the angles F, E^d, it is manifest, that if any two adjoining angles, F, E, of any regular polygon be bisected, the bisecting lines, FO, EO, will meet at the center of an inscribed or circumscribed circle.

COR. 2. Hence, within any regular polygon, a point may be found equally distant from the sides, and also from the angular points, namely, by bisecting any two adjoining sides with perpendiculars, or by bisecting any two adjoining angles. This point is called the *center* of the polygon.

^a Conf. and I. I. ^b Ax. 3. I. ^c 6. I. ^d I. I.

^e Def. 1. 3. and 6. 3.

COR. 3. Hence also it appears, that if from the center, O , of any given regular polygon, $ABCDEF$, the equal distances or perpendiculars OG , OH , OI , &c. be taken, the junction of the points G , H , I , &c. will form a regular polygon, $GHIKLM$, of the same number of sides as the given one. And it is also manifest, that, if from the center, O , of any given regular polygon, $GHIKLM$, lines be drawn to the angular points G , H , I , &c. and, perpendicular to these lines, other lines, AF , FE , ED , &c. be drawn, meeting one another in F , E , &c. a regular polygon, $ABCDEF$, will thus be formed, having the same number of sides as the given one.—Thus may a regular polygon be inscribed in, or described about any given regular polygon, which shall have the same number of sides with the given one.

PROBLEM XXVI.

TO describe a polygon in or about one circle (ABD) which shall be similar to any polygon ($EFGH$) described in or about any other circle (EFH).—(203, 204, 205, 206).

PART I. DRAW the radii QE , QF , &c. (204); at the center P (203) make angle $APB = EQF$, $BPC = FQG$, &c. and join AB , BC , &c. $ABCD$ is similar to $EFGH$. For triangles APB , EQF ;

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BPC, FQG; &c. are mutually equiangular^a: whence, angle $ABC = EFG$, $BCD = FGH$, &c.^b, also $AB :: EF$ ($:: BP : FQ$) $:: BC : FG$ ($:: CP : GQ$) $:: CD : GH$ &c.^c.

PART II. DRAW the perpendiculars QL, QM, &c.^d (206); at the center P (205) make angle $RPS = LQM$, $SPT = MQN$, &c.^e, and draw AB, BC, &c. perpendicular to PR, PS, &c.^f, and meeting one another in B, C, &c. ABCD is similar to EFGH.—For, the quadrangles VR, OL, having three angles equal to three^g, have angle $VAR = OEL$ ^h: and, in a similar manner, angle $RBS = LFM$, &c. Again (drawing PA, PB, &c. QE, QF, &c.) in triangles PAV, PAR (because PV = PR and PA common) angle $PAR = PAV$ ⁱ = $\frac{1}{2} VAR$; and, in like manner, angle $QEL = \frac{1}{2} OEL$: whence, angle $PAB = QEF$ ^b. For similar reasons, angle $PBA = QFE$, angle $PBC = QFG$, &c. Therefore the triangles PAB, QEF; PBC, QFG; &c. are mutually equiangular^k. Consequently $AB : EF$ ($:: BP : FQ$) $:: BC : FG$ &c.^c.

COR. HENCE, similar polygons described in or about circles, are to one another as the squares of the radii, diameters, or circumferences of those circles.—For (in Part 1.) $ABCD : EFGH :: ABsq. : EFsq.$ ^l $:: APsq. : EQsq.$ ^m. And (in Part 2.) $ABCD : EFGH :: ABsq. : EFsq.$ ^l $:: APsq. : EQsq.$ ^m $:: RPsq. : LQsq.$ ^m.

^a Cor. 5. 4. 1. and 6. 1. ^b Ax. 3. 1. ^c 4. 5. ^d 6. 8.

^e 9. 8. ^f 5. 2. ^g Conf. ^h Cor. 6. 4. 1. and Ax. 3. 1.

ⁱ Cor. 12. 1. ^k Cor. 5. 4. 1. ^l 9. 5. ^m 8. 5. and Cor. 8. 5.



